Matematisk modellering og numeriske metoder

Hints to the exercises related to Lecture 9

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Section 11.2

Exercise 1

- To show that a function f is neither even nor odd, it is enough to find one x such that $|f(x)| \neq |f(-x)|$, since if f is even or odd, then $|f(x)| = |f(-x)|$ *for all* x (verify this).
- Remember that *products* of even and odd functions behave as *sums* of even and odd numbers. This means that if a function can naturally be written as the product of even and/or odd functions, we do not have to check the parity of the function itself, but just check the parity of each factor and "add their parities". Here is an example. Let f be given by

  $$f(x) = x^2 \sin(x).$$

  The function $x \mapsto x^2$ is obviously even and $\sin$ is known to be odd. Therefore, f must be odd ("even plus odd equals odd").
- Note that by the definition of $\sinh$ and $\cosh$, $\sinh - \cosh$ can be written as:

  $$\sinh(x) - \cosh(x) = -e^{-x}.$$

  Then use the first point of the hints for this exercise.

Exercise 11

- Unfortunately, I have already accidentally answered the first question about the parity of this function in point two of the hints to Exercise 1.

- By the answer to the first question (see above), the Fourier series is given by the following coefficients:

$$a_0(f) = \frac{1}{1}\int_0^1 x^2 \,\mathrm{d}x \qquad \text{and}$$
$$a_n(f) = \frac{2}{1}\int_0^1 x^2 \cos\Big(\frac{n\pi}{1}x\Big)\,\mathrm{d}x \quad \text{for} \quad n \in \mathbb{N}$$

  (explain to yourselves what is going on here – it has obviously something to do with Theorem 1.1 of Lecture 9 and with changing periods).

- An antiderivative of $x \mapsto x^2 \cos(n\pi x)$ is

$$\frac{(n^2\pi^2x^2 - 2)\sin(n\pi x) + 2n\pi x\cos(n\pi x)}{n^3\pi^3}$$

  (verify by differentiation).

- Remember what $\sin(n\pi x)$ and $\cos(n\pi x)$ evaluates to at 0 and 1.
- You should get $a_0(f) = \frac{1}{3}$ and $a_n(f) = \frac{4}{\pi^2}\frac{(-1)^n}{n^2}$.
- When writing down the series, remember to scale the $\cos$'s correctly!

Exercise 15

- It should be obvious what the parity of the function is.
- The function f is given by

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

  and the parity and periodicity (we do not need to specify f for negative values when we know the parity).

- By the two points above, you need only consider ($\frac{2}{\pi}$ times)

$$\int_0^{\frac{\pi}{2}} x\sin(nx)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\,\mathrm{d}x$$

  (explain this – it has something to do with Theorem 1.1 of Lecture 9).

- Antiderivatives of the two integrants are

$$\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n} \qquad \text{and} \qquad -\frac{\sin(nx)}{n^2} + \frac{x\cos(nx)}{n} - \frac{\pi\cos(nx)}{n}$$

  (find out which function is the antiderivative of which integrant e.g. by differentiation).

- Note that some terms (the $\cos$-terms) cancel each other.
- Others terms appear in both integrals with the same sign.
- A possible term contains $\sin(n\pi)$ which is always 0.
- The last surviving term should contain a $\sin(\frac{n\pi}{2})$ which behaves in the following way:

$$\sin\Big(\frac{n\pi}{2}\Big) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} \\ 1 & \text{if } n \in 4\mathbb{N}-3 \\ -1 & \text{if } n \in 4\mathbb{N}-1 \end{cases}.$$

- All in all, the Fourier series should be given by

$$a_0(f) = a_n(f) = b_{2n}(f) = 0, \qquad b_{4n-3}(f) = \frac{4}{(4n-3)^2\pi}, \qquad \text{and} \qquad b_{4n-1}(f) = -\frac{4}{(4n-1)^2\pi}$$

for all $n \in \mathbb{N}$.

Exercise 19

- Don't feel stupid if you didn't know these identities. The word "familiar" is in my view somewhat of an overstatement.
- There are two ways of seeing this: the easy and the hard way.
- The hard way is to calculate the Fourier series of the functions $x \mapsto \cos^3(x)$ and $x \mapsto \sin^3(x)$ and recognize the results as the identities.
- The easy way is to say: "Hey, if we put $a_1 = \frac{3}{4}$ and $a_3 = \frac{1}{4}$ and all other coefficients equal to 0, then we can write the first identity as a Fourier series! If we put $b_1 = \frac{3}{4}$ and $b_3 = -\frac{1}{4}$ and all other coefficients equal to 0, then we can write the second identity as a Fourier series!"
- By "develop $\cos^4(x)$" they mean: take the function $x \mapsto (\cos(x))^4$ and find its Fourier series!

Exercise 20

- From Exercise 11, you know that $f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2} \frac{(-1)^n}{n^2} \cos(n\pi x)$ pointwise.
- In particular (since $\cos(n\pi) = (-1)^n$),

$$1 = f(1) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2} \frac{(-1)^n}{n^2} \cos(n\pi 1) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- Consider the identity consisting of the first and the last expression in the above identity and isolate the infinite sum.

Exercise 25

- Note that this was all done last time (Exercise 17 and 21 of Section 11.1).
- Repeat your calculations, but where you take advantage of the parity of the function and the fact that you can do with integration over half the interval only.

Exercise 26

- We have the function f given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} & \text{if } \frac{\pi}{2} \leq x \leq \pi \end{cases}. \tag{1}$$

- As usual, we use the formulas found e.g. in Theorem 1.1 of Lecture 9 and split the integral according to (1).
- Antiderivatives for the $\cos$ cases are

$$\frac{\cos(nx)}{n^2} + \frac{x\sin(nx)}{n} \qquad \text{and} \qquad \frac{\pi\sin(nx)}{2n}.$$

  while for the $\sin$ cases, they are (you should know these by now)

$$\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n} \qquad \text{and} \qquad -\frac{\pi\cos(nx)}{2n}$$

- The coefficients of the Fourier cosine series are:

$$\begin{aligned} a_0(f) &= \frac{3\pi}{8}, \\ a_{4n-3}(f) &= -\frac{2}{(4n-3)^2\pi}, \\ a_{4n-2}(f) &= -\frac{1}{(2n-1)^2\pi}, \qquad \text{(note that } 2n-1 = \tfrac{4n-2}{2}\text{)} \\ a_{4n-1}(f) &= -\frac{2}{(4n-1)^2\pi}, \qquad \text{and} \\ a_{4n}(f) &= 0, \end{aligned}$$

  for $n \in \mathbb{N}$.

- The coefficients of the Fourier sine series are:

$$\begin{aligned} b_{2n-1}(f) &= \frac{1}{2n-1} + \frac{2}{(2n-1)^2\pi} \qquad \text{and} \\ b_{2n}(f) &= -\frac{1}{2n}, \end{aligned}$$

  for $n \in \mathbb{N}$.

Exercise 27

- The function is now f given by

$$f(x) = \begin{cases} \frac{\pi}{2} & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

- Antiderivatives can be found in the hints for Exercise 15 and 26.
- Apart from that, look at the hints for Exercise 26 and Exercise 30 for more details.

Exercise 30

- Note first that if we call the function in Exercise 26 for f_{26} and the function in Exercise 27 for f_{27}, denote even extensions of a function f by f^e and odd extensions of a function f by f^o, then:

$$f^e_{27}(x) = f^e_{26}(x - \pi) \qquad \text{and} \qquad f^o_{27}(x) = -f_{26}(x - \pi) \tag{2}$$

(make a sketch to convince yourselves or prove it from the formulas).

- Since

$$f^e_{26}(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty} a_n(f^e_{26}) \cos(nx),$$

we get that

$$f^e_{27}(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty} a_n(f^e_{26}) \cos(n(x - \pi)). \tag{3}$$

- Since $\cos(n(x-\pi)) = \cos(nx - n\pi) = (-1)^n \cos(nx)$, (3) becomes

$$f^e_{27}(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty} (-1)^n a_n(f^e_{26}) \cos(nx),$$

so the Fourier coefficients of f^e_{27} are given by

$$a_0(f^e_{27}) = a_0(f^e_{26}) \qquad \text{and} \qquad a_n(f^e_{27}) = (-1)^n a_n(f^e_{26}).$$

- By a similar argument, we get

$$b_n(f^o_{27}) = -(-1)^n b_n(f^o_{26}),$$

where the extra -1 in front of the right-hand side comes from a similar sign in (2).