Matematisk modellering og numeriske metoder

Lecture 10

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1 Partial Differential Equations

1.1 The Basics of PDE's

[Section 12.1 in the book, p. 540]

Definition 1.1 (Partial Differential Equations (PDE's))**.** An equation in an unknown function u of several variables is called a *partial differential equation (PDE)*, if the equation depends on one or more partial derivatives of u. The order of the highest partial derivative is called the *order* of the PDE.

The unknown function will usually be considered a function of one or several space variables and often of one time variable. As was the case with ODE's, the most important PDE's in applications will be of second order.

Remember that partial derivation is linear in the function (i.e. $\frac{\partial cu}{\partial x} = c\frac{\partial u}{\partial x}$ for all constants c, if $u = u_1 + u_2$, then $\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}$, etc.). This makes the following definition natural.

Definition 1.2 (Linear PDE's, homogeneity)**.** A PDE in the variables $x_1, \dots x_n$ of order k is called *linear* if it can be brought to the form

$$F\left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots, \frac{\partial^k u}{\partial x_1^k}, \dots, \frac{\partial^k u}{\partial x_1 \partial x_2 \cdots \partial x_n}, \dots, \frac{\partial^k u}{\partial x_n^k}\right) = r$$

for some linear function F and some function r of the independent variables $x_1, \dots, x_n$. If for a linear PDE, r can be chosen to be 0, the PDE is called *homogenous*, otherwise it is *non-homogenous*.

Note that in the above definition, the independent variables are just given the *names* $x_1, \dots, x_n$ in order to be able to refer to them. There is no indication of them being spatial variables and if a particular PDE depends on time t, then this variable should also be included in the list of independent variables and used for partial derivations. In particular, a linear PDE of order two where the unknown function u depends on one spatial variable x and one temporal variable t is of the form:

$$F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}\right) = r,$$

where F is linear and r is a function of x and t. Such PDE's are often referred to as *one-dimensional* in spite of the fact that the unknown function depends on *two* variables, since it depends on a one-dimensional spatial variable.

Example 1.3. The following six examples of second order linear PDE's are important in many applications.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(one-dimensional wave equation)} \tag{1}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(one-dimensional heat equation)} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(two-dimensional Laplace equation)} \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \qquad \text{(two-dimensional Poisson equation)} \tag{4}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{(two-dimensional wave equation)} \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{(three-dimensional Laplace equation)} \tag{6}$$

All but the Poisson equation (4) are homogeneous, all "dimensions" mentioned are spatial, the wave and heat equations (1), (2), (5) depend also on time t, while the Laplace and Poisson equations are purely spatial (*time-independent*).

Definition 1.4 (Solution of a PDE)**.** A *solution* to a PDE in some region R of the space of independent variables (including the possible temporal variable t) is a function u for which all partial derivates appearing in the PDE exist in some domain[1] containing R such that u satisfies the PDE in R.

As in IVP's in ODE's, where we needed initial conditions to be sure that the solution was the right one, we will need *additional conditions* for PDE's. These are typically given as *boundary conditions* (the solution is required to have certain values at the boundary of the region R) and/or (if time t is an independent variable) *initial conditions* (the value of u at time $t = 0$ and/or of one or several partial derivatives of u with respect to t at time $t = 0$ is required to have a certain value).

These additional conditions are even more important for PDE's than for ODE's, as the increased complexity of PDE's increases the space of solutions dramatically. A fairly simple second order PDE such as the two-dimensional Laplace equation (3) has, for example, the following very different solutions:

$$u(x, y) = x^2 - y^2, \quad u(x, y) = e^x \cos(y), \quad u(x, y) = \sin(x) \cosh(y), \quad \text{and} \quad u(x, y) = \ln(x^2 + y^2).$$

Since (3) is obviously homogeneous, you have probably guessed that also linear combinations of these solutions are solutions, as stated in the following theorem:

[1] a domain is a connected open set.

Theorem 1.5 (Theorem 1 on page 541 in the book)**.** *The space of solutions to a homogeneous PDE in some region R is closed under linear combinations, i.e. if u_1 and u_2 are solutions on R, then so are*

$$u = c_1 u_1 + c_2 u_2$$

for any choice of c_1 and c_2.

The proof is in spirit no different from the ODE version of the same statement.

We will now consider some very simple PDE's that are solvable by ODE methods. But first a comment on notation: it can be rather tiresome to write all those ∂'s. To remedy this, a shorter notation for partial derivates has been invented. Let u be a function. The very simple idea is to write partial derivatives of u with respect to a variable x by writing the x as an index: u_x. As $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ if one (and hence both) of these expressions is continuous, we may use this notation rule successively and write e.g. $u_{xy} = u_{yx}$ or u_{xx}.

Example 1.6 (Example 2 on page 542 in the book)**.** We consider the PDE

$$u_{xx}(x, y) - u(x, y) = 0.$$

Had it not been for the y-dependence, this would just be $u'' - u = 0$ and the solution would be $u(x) = ae^x + be^{-x}$ for any constants a, b. This means that for any fixed choice of y_0, $a = a_{y_0}$, and $b = b_{y_0}$, the function $u(x, y_0) = a_{y_0} e^x + b_{y_0} e^{-x}$ satisfies the PDE (but is not a solution, as the set $\{(x, y_0) : x \in \mathbb{R}\}$ is not an open set). This means that for arbitrary functions $a, b \colon \mathbb{R} \to \mathbb{R}$,

$$u(x, y) = a(y)e^x + b(y)e^{-x}$$

is a solution to the PDE.

Example 1.7. We consider the PDE $u_{xy} = -u_x$. First, write $p(x, y) = u_x(x, y)$. Then the PDE can be written as

$$p_y = -p. \tag{7}$$

Had it not been for the x-dependence, this would just be $p' = -p$, and the solution would be $p(y) = ce^{-y}$ for any constant c. This means that for any fixed choice of x_0 and $c = c_{x_0}$, the function $f(x_0, y) = c_{x_0} e^{-y}$ satisfies (7). As before, this means that for an arbitrary function $c \colon \mathbb{R} \to \mathbb{R}$, $f(x, y) = c(x)e^{-y}$ satisfies (7). To get to u, we just have to integrate with respect to x:

$$u(x, y) = f(x)e^{-y} + g(y), \qquad \text{where} \qquad f = \int c(x)\,\mathrm{d}x,$$

and g is an integration constant which may depend on y. Since $c(x)$ and $g(y)$ can be chosen freely, so can f (as long as it is differentiable), and the solution is just

$$u(x, y) = f(x)e^{-y} + g(y)$$

for arbitrary (but differentiable) f and g.

Verify the solutions above by direct computation.

1.2 Derivation of the wave equation

[Section 12.2 in the book, p. 543]
In the previous section, we saw a couple of so-called wave equations. We will now derive the one-dimensional wave equation as a model for a vibrating string. The setup is as follows. We have a string along the x-axis stretched to length L and fixed at the endpoints $x = 0$ and $x = L$. The string is distorted and released at time $t = 0$ to vibrate freely. We want to model its deflection $u(x, t)$ at any point $x \in [0, L]$ and for any time $t > 0$. To set up the model, we make the following assumptions.

1. The string has uniform mass along the x-axis; we let ρ denote the mass per unit length.
2. The string is perfectly elastic and does not offer any resistance to bending.
3. The gravitational force acting on the string can be neglected.
4. Each particle of the string only moves in the y-direction, perpendicular to the x-axis along which it is stretched.

We consider the forces acting on a small portion of the string. Let x be some point and $x+\Delta x$ some nearby point on the x-axis, with P and Q being the endpoints of the deflected string between x and $x + \Delta x$. Because of the assumptions, the only forces acting on the string are the tangential tensions at each point. Let T_1 and T_2 denote the size of the tensions at P and Q, respectively. If we denote by α resp. β the angles between the tangents at P resp. Q and the x-axis, then, since there is no motion along the x-axis, the two forces along the x-direction must be of equal size:

$$T_1 \cos(\alpha) = T_2 \cos(\beta) = T$$

In the y-direction, we have two forces, namely $-T_1 \sin(\alpha)$ and $T_2 \sin(\beta)$, where the minus sign appears because the force at P is in the negative y-direction. By Newton's second law and the mean value theorem, the resultant of the two forces is $\rho\Delta(x)\frac{\partial^2 u}{\partial t^2}(x', t)$ for some $x' \in [x, x + \Delta x]$. Hence

$$T_2 \sin(\beta) - T_1 \sin(\alpha) = \rho\Delta(x)\frac{\partial^2 u}{\partial t^2}(x', t) \tag{8}$$

Dividing (8) by T gives

$$\frac{T_2 \sin(\beta)}{T_2 \cos(\beta)} - \frac{T_1 \sin(\alpha)}{T_1 \sin(\alpha)} = \tan(\beta) - \tan(\alpha) = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}(x', t).$$

Noting that $\tan(\alpha) = \frac{\partial u}{\partial x}(x, t)$ and $\tan(\beta) = \frac{\partial u}{\partial x}(x + \Delta x, t)$ and dividing by Δx yields:

$$\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right) = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}(x', t),$$

where the left-hand side tends to $\frac{\partial^2 u}{\partial x^2}(x, t)$ while the right-hand side tends to $c^2\frac{\partial^2 u}{\partial t^2}(x, t)$ where we have written $c^2 = \frac{T}{\rho}$ as a square to indicate its positivity (which is important for the class of solutions). We have thus derived the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}(x, t) = c^2\frac{\partial^2 u}{\partial t^2}(x, t) \tag{9}$$

which is a homogeneous second order PDE.

1.3 Solution of the wave equation

[Section 12.3 on page 545 of the book]
The problem of the vibrating string is not completely specified by (9) without the following two *additional conditions*. Our string is fixed at the endpoints, so for all times $t \geq 0$, we require the *boundary conditions*

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0. \tag{10}$$

When we distorted and released the string, we did it in a certain manner. More precisely, it had the following *initial conditions* for all $x \in [0, L]$

$$u(x,0) = f(x) \qquad \text{and} \qquad u_t(x,0) = g(x), \tag{11}$$

where f is the initial deflection and g is the initial velocity of the string. We will now solve the PDE (9) under the additional conditions (10) and (11). This will be done in three steps.

Step 1: The method of separating variables or the product method

We assume that the solution can be written on the form $u(x,t) = F(x)G(t)$ for some functions F and G. If this is the case, then

$$\frac{\partial^2 u}{\partial t^2} = FG'' \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = F''G.$$

Inserting this into the wave equation and dividing by c^2FG, we get

$$\frac{G''}{c^2G} = \frac{FG''}{c^2FG} = \frac{c^2F''G}{c^2FG} = \frac{F''}{F},$$

where the leftmost expression only depends on t while the rightmost only depends on x. They must therefore be constant. Equating both expressions with the constant k yields after rearranging

$$F'' - kF = 0 \qquad \text{and} \qquad G'' - c^2kG = 0,$$

which are two ODE's.

Step 2: Finding solutions that satisfy the boundary conditions

If $G \not\equiv 0$, then we need $F(0) = 0$ and $F(L) = 0$ in order to satisfy (10). We now use our knowledge of ODE's to conclude that k must be negative; otherwise $k = 0$ in which case $F(x) = ax + b$, or $k > 0$ in which case $F(x) = ae^{\sqrt{k}x} + be^{-\sqrt{k}x}$, but in both cases $F(0) = F(L) = 0$ then implies that $F \equiv 0$. Hence if $u \not\equiv 0$, then $k < 0$ and

$$F(x) = a\cos(px) + b\sin(px),$$

where p is given by $k = -p^2$. But $F(0) = F(L) = 0$ implies that $a = 0$ and $b\sin(pL) = 0$, so $\sin(pL) = 0$ since $b = 0$ would imply that $F \equiv u \equiv 0$. In other words,

$$p = \frac{n\pi}{L}, \qquad \text{for} \qquad n \in \mathbb{Z}.$$

Setting $b = 1$ we thus obtain infinitely (countably) many solutions $F_n(x) = \sin(\frac{n\pi}{L}x)$ for $n \in \mathbb{N}$ ($n = 0$ is $F \equiv 0$ and $\mathbb{Z} \ni n < 0$ amounts to choosing $b = -1$ as $\sin(-x) = -\sin(x)$).

That was F. We now solve for G, keeping in mind that $k = -p^2 = -(\frac{n\pi}{L})^2$, so

$$G'' + \lambda_n^2 G = 0, \qquad \text{where} \qquad \lambda_n = cp = \frac{cn\pi}{L}.$$

From our ODE knowledge, we recall that

$$G_n(t) = b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)$$

is a general solution. We have thus found solutions to the original problem:

$$u_n(x,t) = (b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)) \sin\Big(\frac{n\pi}{L}x\Big).$$

These are called *eigenfunctions* with *eigenvalues* λ_n of the vibrating string. The set of eigenvalues is called the *spectrum*. The motions represented by the eigenfunctions are called the n'th *normal mode* and have *frequency* $\frac{\lambda_n}{2\pi}$, the first normal mode is called the *fundamental mode*, and the others are called *overtones*. Since

$$\sin\Big(\frac{n\pi x}{L}\Big) = 0 \qquad \text{at} \qquad x = \frac{L}{n}, \frac{2L}{n}, \ldots, \frac{n-1}{n}L,$$

the n'th normal mode has $n-1$ *nodes*, i.e. points $x \in (0, L)$ that are constantly 0. We also note that the frequency $\frac{\lambda_n}{2\pi} = \frac{cn}{2L} = \frac{\sqrt{\frac{T}{\rho}}n}{2L}$ of u_n grows with the tension T and decreases with the length L.

Step 3: Finding solutions that also safisfy the initial conditions

So far, we have constructed solutions which satisfy the PDE (9) and the boundary conditions (10). Apart from some special cases, we have yet to find a solution that satisfies the initial conditions (11). The idea is now to take linear combinations of the u_n in such a way that the resulting function satisfies the initial conditions. We know from Theorem 1.5 that we may take (finite) linear combinations of solutions. We now take the chance and try with an *infinite* linear combination:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t)) \sin\Big(\frac{n\pi}{L}x\Big). \tag{12}$$

In order for the initial displacement f (11) to match our guess (12), we need the following to hold:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi}{L}x\Big) = f(x).$$

this means that the b_n should be the Fourier coefficients of the odd half range expansion of f,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x.$$

Assuming we can differentiate termwise in the infinite sum, we similarly get for g that

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n^* \lambda_n \sin\Big(\frac{n\pi}{L}x\Big) = g(x).$$

This amounts to setting

$$b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin\Big(\frac{n\pi}{L} x\Big)\, \mathrm{d}x.$$

All in all, if the series converges and we can differentiate termwise, then we have just established a solution that satisfies all additional conditions (10) and (11). We will now in the case where $g \equiv 0$ (no differentiation of the infinite sum is needed!) argue that the series in fact converges for sufficiently nice f. Then our candidate reduces to

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(\lambda_n t) \sin\Big(\frac{n\pi}{L} x\Big), \qquad \text{where} \qquad \lambda_n = \frac{cn\pi}{L}.$$

We now use the addition formula:

$$\cos\Big(\frac{cn\pi}{L} t\Big) \sin\Big(\frac{n\pi}{L} x\Big) = \frac{1}{2}\bigg(\sin\Big(\frac{n\pi}{L}(x - ct)\Big) + \sin\Big(\frac{n\pi}{L}(x + ct)\Big)\bigg).$$

This means that

$$u(x,t) = \sum_{n=1}^{\infty} b_n \frac{1}{2}\bigg(\sin\Big(\frac{n\pi}{L}(x - ct)\Big) + \sin\Big(\frac{n\pi}{L}(x + ct)\Big)\bigg).$$

Write f^* for the odd, $2L$-periodic extension of f. Then, for sufficiently nice f,

$$f^*(x \pm ct) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi}{L}(x \pm ct)\Big)$$

pointwise. In fact one can prove (again for sufficiently nice f), that one can interchange the order of the sum, so that

$$u(x,t) = \frac{1}{2}(f^*(x - ct) + f^*(x + ct)). \tag{13}$$

A very natural assumption is that f is continuous (otherwise the string is "broken" at time $t = 0$), and this is more than enough to be sufficiently nice. Cf. also Theorem 1.5 of Lecture 8. If we now assume that f is twice differentiable on $(0, L)$ and has one-sided second derivatives at $x = 0$ and $x = L$ which are zero, then (13) can be seen to satisfy (9), (10) and (11) for $g \equiv 0$ by direct computation.

If f' and f'' are merely piecewise continuous, or the one-sided derivatives are not 0, then for each t there will be finitely many x for which the second derivatives of (9) do not exist. However, apart from at these points, the wave equation still holds. In this case the solution (13) is called a *generalized solution*.