Matematisk modellering og numeriske metoder

Lecture 2

Morten Grud Rasmussen

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1 Laplace Transforms

1.1 The definition of the Laplace transform

Definition 1.1 (Laplace transform)**.** Let f be a function defined on the set of non-negative reals. If the integral

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \tag{1}$$

exists, then we call $\mathcal{L}(f)$ the *Laplace transform* of f. The *linear operator* $\mathcal{L}$ that takes a function into its Laplace transform is also called the Laplace transform.

Remark 1.2. The notation $\int_0^\infty f(t)\,\mathrm{d}t = I$ means that the quantity $\int_0^M f(t)\,\mathrm{d}t = I(M)$ exists for all $M > 0$ and that the limit $\lim_{M\to\infty} I(M)$ exists and equals I.

The Laplace transform of a function denoted by a lower-case letter is often written as the same letter in upper-case form. We will reserve the letter t for the independent variable of the original function while s is the independent variable of the Laplace transform. The (linear operator) Laplace transform $\mathcal{L}$ is a so-called *integral transform*

$$\mathcal{L}(f)(s) = \int_{-\infty}^\infty k(s,t) f(t)\,\mathrm{d}t$$

with *kernel* $k(s,t) = e^{-st}$ for $t \geq 0$, $k(s,t) = 0$ otherwise. Integral transforms are characterized by tranforming functions from one function space into another function space by means of integrating the function with a kernel.

If $F(s) = \mathcal{L}(f)(s)$ then we write

$$f = \mathcal{L}^{-1}(F).$$

Since the integral in (1) doesn't change if we change the value of f in finitely many places, strictly speaking $\mathcal{L}^{-1}(F)$ is not uniquely defined. However, this is of little practical importance and can be avoided either by grouping functions into *equivalence classes* or, perhaps more relevant for our purposes, by only considering continuous functions (continuous functions cannot be changed in finitely many places and remain continuous).

Example 1.3. Let $f(t) = 1$ for $t \geq 0$. We want to find $\mathcal{L}(f)$. By definition,

$$\mathcal{L}(f)(s) = \mathcal{L}(1)(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = -\frac{1}{s}e^{-st}\bigg|_{t=0}^{\infty} = \frac{1}{s},$$

whenever $s \neq 0$.

Example 1.4. Let $f(t) = e^{at}$ for $t \geq 0$ and some constant a. We want to find $\mathcal{L}(f)$. Again, we use the definition:

$$\mathcal{L}(e^{at})(s) = \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t = -\frac{1}{s-a}e^{-(s-a)t}\bigg|_{t=0}^{\infty} = \frac{1}{s-a},$$

whenever $s - a \neq 0$.

1.2 Linearity of the Laplace transform

To avoid having to calculate the Laplace transform of a function by using the definition every time, we will examine some of the general properties of the Laplace transform. The first property has already been mentioned, but we will now formally define it and prove that it in fact holds for the Laplace transform.

Definition 1.5 (Linear operator)**.** A *linear operator* L is a mapping from one vector space X to another vector space Y that preserves the linear structure, i.e. $L(ax + by) = aL(x) + bL(y)$ for all vectors $x, y \in X$ and all scalars a, b.

We stress that, although we haven't precisely stated what the initial and final function spaces for the Laplace transform are, they are indeed vector spaces, and this is all we need to know for now. The claim of last section is now contained in the following theorem.

Theorem 1.6 (The Laplace transform is a linear operator)**.** *The Laplace transform is a linear operator.*

Proof. Let a and b be real numbers, and assume that f and g lie in the domain of $\mathcal{L}$, that is, $\int_0^\infty e^{-st}f(t)\,\mathrm{d}t$ and $\int_0^\infty e^{-st}g(t)\,\mathrm{d}t$ exist. This means that

$$I_f(M) = \int_0^M e^{-st}f(t)\,\mathrm{d}t \quad \text{and} \quad I_g(M) = \int_0^M e^{-st}g(t)\,\mathrm{d}t$$

exist for all M and have finite limits

$$\lim_{M\to\infty} I_f(M) = I_f = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t \quad \text{and} \lim_{M\to\infty} I_g(M) = I_g = \int_0^\infty e^{-st}g(t)\,\mathrm{d}t.$$

But, since integration is linear, we must have

$$\int_0^M e^{-st}(af(t) + bg(t))\,\mathrm{d}t = a\int_0^M e^{-st}f(t)\,\mathrm{d}t + b\int_0^M e^{-st}g(t)\,\mathrm{d}t = aI_f(M) + bI_g(M) \qquad (2)$$

where the right-hand-side clearly converges to $aI_f + bI_g$. Since (2) holds true for all M, the left-hand-side has the same limit and we conclude

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$$

for all a, b, f and g. $\square$

Example 1.7. We want to find the Laplace transform of $f\colon t \mapsto \cosh(at)$. Since

$$\cosh(at) = \frac{1}{2}(e^{at} + e^{-at}),$$

we can take advantage of the linearity and Example 1.4 and obtain

$$\mathcal{L}(f)(s) = \frac{1}{2}(\mathcal{L}(e^{a\cdot}) + \mathcal{L}(e^{-a\cdot}))(s) = \frac{1}{2}\Big(\frac{1}{s-a} + \frac{1}{s+a}\Big) = \frac{s}{s^2 - a^2},$$

where $e^{a\cdot}$ denotes the function $t \mapsto e^{at}$ (mathematicians don't write the t because there is no t dependence).

The Laplace transform of $t \mapsto \sinh(at)$ can be calculated similarly while the non-hyperbolic versions with our present knowledge needs a trick to be calculated (see Example 4 on page 206 in the book), although one could be tempted to cheat and use the so-called Euler representations $\cos(at) = \frac{1}{2}(e^{iat} + e^{-iat})$ and $\sin(at) = \frac{1}{2i}(e^{iat} - e^{-iat})$ (ignoring that the exponent is complex and just using the previous method algebraically, one would get the right result, however, we have only defined the Laplace transform for real-valued functions and we have not yet seen how to handle complex integrals).

1.3 The Laplace tranform of polynomials

By linearity, the section heading just above can easily be reduced to "The Laplace transform of simple monomials" i.e. functions of the form $f\colon t \mapsto t^n$ for some natural number n. The formula for $\mathcal{L}(f)$ in this case is derived on page 207 in the book, but we will instead derive the more general formula that for $f\colon t \mapsto t^a$, we have

$$\mathcal{L}(f)(s) = \frac{\Gamma(a+1)}{s^{a+1}}.$$

(This formula is also derived in the book; see page 208.) Since $\int_0^\infty e^{-x}x^n \,\mathrm{d}x = \Gamma(n+1) = n!$, this reduces to the other formula whenever $a = n$ is a natural number. Note however that we are somewhat cheating in that we leave out the proof of $\Gamma(n+1) = n!$. The derivation goes as follows.

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}t^a \,\mathrm{d}t = \int_0^\infty e^{-x}\Big(\frac{x}{s}\Big)^a \frac{\mathrm{d}x}{s} = \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a \,\mathrm{d}x = \frac{\Gamma(a+1)}{s^{a+1}}$$

where we used substitution $st = x$ and the defining formula for $\Gamma(a+1)$.

1.4 Replacing s by $s-a$ in the transform

The act described in the heading of this section is also referred to as *s-shifting* and is done using the following simple fact.

Theorem 1.8. *If f has the transform $s \mapsto F(s)$ for $s > k$, where k is some sufficiently large number, then $t \mapsto e^{at} f(t)$ has the transform $s \mapsto F(s-a)$ for $s - a > k$. In formula:*

$$\mathcal{L}(e^{a\cdot} f)(s) = F(s-a)$$

Proof. let k be such that $F(s)$ exists for $s > k$. Then $F(s-a)$ exists for $s - a > k$ and

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t = \int_0^\infty e^{-st} e^{at} f(t)\,\mathrm{d}t = \mathcal{L}(e^{a\cdot} f)$$

as stated. □

1.5 Existence and uniqueness

So far, we have mostly just *assumed* that the integral existed, although in a few places, we have calculated the integral for concrete functions for which the integral clearly existed. We will now give a general sufficient condition for the integral to exist in terms of the function f we are taking the Laplace transform of.

Theorem 1.9. *Let f be integrable on any finite interval (on the positive half-axis) and have growth of at most exponential order:*

$$|f(t)| \leq M e^{kt}$$

for some constants M and k. Then the Laplace transform $s \mapsto \mathcal{L}(f)(s)$ exists for all $s > k$.

We note that a sufficient condition for being integrable on any finite interval is to be (piecewise) continuous.

Proof. Assume that $s > k$. Then

$$\frac{M}{s-k} = \int_0^\infty M e^{kt} e^{-st}\,\mathrm{d}t \geq \int_0^\infty |f(t)| e^{-st} \geq \Big| \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \Big| = |\mathcal{L}(f)(s)|.$$

Since the first number is finite, so is the last. □

This was existence. What about uniqueness? We have already touched this point; $\mathcal{L}(f)$ is in general not "born" unique, in the sense that $\mathcal{L}(f) = \mathcal{L}(\tilde{f})$ for (slightly) different f and $\tilde{f}$, but this is not more severe than it can be handled by e.g. restricting to continuous functions. In general, one can in fact make a mathematically rigorous statement saying in principle that if $\mathcal{L}(f) = \mathcal{L}(\tilde{f})$, then f and $\tilde{f}$ are *essentially* identical.

2 The Laplace transform and ODE's

So, what is the point with these Laplace transforms? Well, it turns out that the Laplace transform provides a means of transforming an IVP into an algebraic problem, whose solution can be transformed back into a solution of the IVP. A basic ingredient is the following.

2.1 Laplace transforms of derivatives

Theorem 2.1 (The laplace transform of the n'th derivate of a function)**.** *Assume that the k'th derivative $t \mapsto f^{(k)}(t)$ of a function f is continuous for all $t \geq 0$ and grows at most exponentially for all $k \leq n-1$. Assume that $f^{(n)}$ is piecewise continuous on every finite interval of the half-axis. Then*

$$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

for all sufficiently large s. In particular, for $n = 1$ and $n = 2$, respectively, we have

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$$

and

$$\mathcal{L}(f'')(s) = s^2 \mathcal{L}(f)(s) - sf(0) - f'(0).$$

Proof. We begin by showing the case $n = 1$. Assume first that f' is continuous (not just piecewise continuous). Then, using the definition and integration by parts, we get

$$\mathcal{L}(f')(s) = \int_0^\infty e^{-st} f'(t) \, \mathrm{d}t = [e^{-st} f(t)] \Big|_{t=0}^{\infty} + s \int_0^\infty e^{-st} f(t) \, \mathrm{d}t \, .$$

Now, by the assumptions, $t \mapsto e^{-st} f(t)$ "evaluated at" ∞ (strictly speaking, one should take the limit) is 0 whenever s is sufficiently large ($s > k$ where k is the k from the growth estimate), while $e^{-s \cdot 0} f(0) = f(0)$ and the last integral is $s\mathcal{L}(f)(s)$. This gives

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0). \tag{3}$$

If f' is only piecewise continuous, the same arguments work on each of the continuous pieces, and by linearity, the conclusion remains the same. The general result now follows by applying (3) to $f^{(n)}$ iteratively:

$$\begin{aligned} \mathcal{L}(f^{(n)})(s) &= s\mathcal{L}(f^{(n-1)})(s) - f^{(n-1)}(0) \\ &= s^2 \mathcal{L}(f^{(n-2)})(s) - sf^{(n-2)}(0) - f^{(n-1)}(0) \\ &\dots \\ &= s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0) \end{aligned}$$

and we're done. □

Example 2.2. Earlier, we have talked about cheating in the calculation of the Laplace transform of $t \mapsto \cos(at)$. We will now do it properly. Let $f(t) = \cos(at)$. Then $f(0) = 1$, $f'(0) = 0$ and $f''(t) = -a^2 f(t)$. Then by linearity and Theorem 2.1, we get two different ways of expressing $\mathcal{L}(f'')$ in terms of $\mathcal{L}(f)$:

$$\mathcal{L}(f'')(s) = -a^2 \mathcal{L}(f)(s) = s^2 \mathcal{L}(f)(s) - s \, .$$

If we now isolate $\mathcal{L}(f)(s)$ in the last equality, we get

$$\mathcal{L}(f)(s) = \frac{s}{s^2 + a^2}.$$

A similar argument gives the Laplace transform of $t \mapsto \sin(at)$. (Hint: Look at $\mathcal{L}(f')$.)

2.2 The Laplace transform of the integral of a function

Theorem 2.3 (The Laplace transform of an integral)**.** *Let F denote the Laplace transform of a piecewise continuous function f of at most exponential growth (where we denote the factor in the exponent by k). Then, for $s > \max(0, k)$ and $t > 0$, we have*

$$\mathcal{L}\Big(\int_0^{\cdot} f(t)\,\mathrm{d}t\Big)(s) = \frac{1}{s}F(s),$$

where $\int_0^{\cdot} f(t)\,\mathrm{d}t$ is the function $\tau \mapsto \int_0^{\tau} f(t)\,\mathrm{d}t$.

Proof. We begin by noting that

$$\Big|\int_0^{\tau} f(t)\,\mathrm{d}t\Big| \leq \int_0^{\tau} |f(t)|\,\mathrm{d}t \leq M\int_0^{\tau} e^{kt}\,\mathrm{d}t = \frac{M}{k}(e^{k\tau} - 1) \leq \frac{M}{k}e^{k\tau}$$

so $\tau \mapsto \int_0^{\tau} f(t)\,\mathrm{d}t$ is of at most exponential growth. Furthermore, $\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\tau} f(t)\,\mathrm{d}t = f(\tau)$ is piecewise continuous and $\int_0^{0} f(t)\,\mathrm{d}t = 0$. We can now apply Theorem 2.1 to f and get

$$\mathcal{L}(f)(s) = s\mathcal{L}\Big(\int_0^{\cdot} f(t)\,\mathrm{d}t\Big)(s).$$

dividing both sides with s now yields the result. □

2.3 The Laplace transform as a tool for solving IVP's

Consider the second-order IVP

$$y''(t) + ay'(t) + by(t) = r(t), \quad y(0) = K_0, \quad y'(0) = K_1,$$

(note that second-order IVP's need not just "$y(x_0) = y_0$" but also "$y'(x_0) = z_0$" to be fully determined). Here, a, b, K_0 and K_1 are constants, and the function r is called the *input* or *driving force* and y is called the *output* or the *response to the driving force*.

The idea is now to take the Laplace transform on both sides of the equation:

$$\mathcal{L}(y'' + ay' + by)(s) = \mathcal{L}(r)(t)$$

which we see can be written as

$$(s^2Y(s) - sy(0) - y'(0)) + a(sY(s) - y(0)) + bY(s) = (s^2 + as + b)Y(s) - (s + a)y(0) - y'(0) = R(s)$$

where $Y = \mathcal{L}(y)$ and $R = \mathcal{L}(r)$. Isolating $Y(s)$ gives

$$Y(s) = \frac{(s + a)y(0) + y'(0) + R(s)}{s^2 + as + b} = \big((s + a)y(0) + y'(0)\big)Q(s) + R(s)Q(s), \tag{4}$$

where $Q(s) = \frac{1}{s^2+as+b} = \frac{1}{(s+\frac{1}{2}a)^2+b-\frac{1}{4}a^2}$ is called the *transfer function*. If $y(0) = y'(0) = 0$ then $Q = \frac{Y}{R} = \frac{\mathcal{L}(y)}{\mathcal{L}(r)}$, which explains the name. Note that Q *depends neither or $r(t)$ nor on the initial conditions* but just on a and b.

Since the solution y is differentiable, it is continuous, and hence the inverse Laplace transform of its Laplace transform (if it exists) is unique. This means that we just need to take the inverse Laplace transform of the algebraic solution to the Laplace transform of the ODE (4) to find a solution to the ODE. This is usually done by rewriting the right-hand-side of (4) as a sum of terms whose inverses can be found in tables or using some computer software.

Example 2.4. We want to solve the second-order IVP

$$y''(t) - y(t) = t, \quad y(0) = 1, \quad y'(0) = 1.$$

We note that $Q(s) = \frac{1}{s^2-1}$ and as $r(t) = t$, $\mathcal{L}(r) = \frac{1}{s^2}$ so (4) becomes

$$Y(s) = \big((s+0)\cdot 1 + 1\big)\frac{1}{s^2-1} + \frac{1}{s^2}\frac{1}{s^2-1} = \frac{1}{s-1} + \Big(\frac{1}{s^2-1} - \frac{1}{s^2}\Big).$$

Now we are ready to put everything together to find the solution to the original problem.

We recognize the first term as an "s-shifted" $\frac{1}{s}$ (shifted by 1). As $\frac{1}{s}$ is the Laplace transform of the constant function 1, and as s-shifts are made by multiplying with the function $t \mapsto e^{at}$ where a is the size of the shift, we conclude that $\mathcal{L}^{-1}(\frac{1}{\cdot-1})(t) = e^t$. The next term, $\frac{1}{s^2-1}$ is the Laplace transform of $t \mapsto \sinh(t)$. Hence $\mathcal{L}^{-1}(\frac{1}{\cdot^2-1})(t) = \sinh(t)$. The last term is the Laplace transform of t.

Summing up,

$$y(t) = e^t + \sinh(t) - t,$$

and we note that this was solved without first finding a general solution.