Matematisk modellering og numeriske metoder

Lecture 7

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1 Divergence of a vector field

[Section 9.8 in the book, p. 403]

1.1 Definition and properties of divergence

Let $v = \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right)$ be a 3-dimensional vector function with continuous partial derivatives which depends on the Cartesian coordinates x, y, and z, and may or may not depend on time. This means that

$$\frac{\partial v_1}{\partial x}, \quad \frac{\partial v_2}{\partial y}, \quad \text{and} \quad \frac{\partial v_3}{\partial z}$$

exist and are continuous functions. You may think of v as the velocity of e.g. air at a fixed time. The partial derivatives then measures the change of speed in each of the directions x, y, and z, not as time changes, but as one moves around in space!

Now, imagine that the described system is confined in a long tube along the x-axis and that the tube has a very small radius. Assume you have a drift along the x-axis in positive direction. If $\frac{\partial v_1}{\partial x}$ is negative, then, as you walk along the x-axis in positive direction, the velocity in the x-direction is reduced. If this picture stays the same for all times, the "ingoing speed" is larger than the "outgoing speed" along this axis!

On the other hand, if $\frac{\partial v_1}{\partial x}$ is negative but we instead have a drift in the negative direction, then, as you walk along the x-axis in the positive direction, the *absolute value* of the velocity in the x-direction actually *increases* (we have that $\frac{\partial v_1}{\partial x} < 0$ so there is a negative slope, but the value was negative, so it gets farther away from the origin). Again, this means that the "ingoing speed" is larger than the "outgoing speed" along the x-axis, the in- and outgoing directions have just swapped places!

We conclude that if $\frac{\partial v_1}{\partial x}$ is negative, then the "ingoing speed" is larger than the "outgoing speed", at least in the x-direction, irrelevant of whether things are coming from "the right" or "the left." The opposite is obviously true if we have $\frac{\partial v_1}{\partial x} > 0$. If the "ingoing speed" is larger than the "outgoing speed" along a certain axis, then one of two things must happen: either this imbalance is counterweighted by the "in- and outgoing speeds" along a complementary axis, or things get compressed.

If the tube is really narrow (small radius), one would expect that only the x-axis matters, so here the two cases above may be thought of as this: either there is a hole in the tube through which the air can slip out (a "*sink*"), or the air really is compressed inside the tube. Had we considered the opposite case ($\frac{\partial v_1}{\partial x} > 0$), the conclusions should be reversed: either air is coming in from outside the tube (a "*source*"), or the air dilutes.

We now have an intuition about what $\frac{\partial v_1}{\partial x}$ tells us about a vector field, namely, it measures the difference between the x-directional *inflow* and the x-directional *outflow*. It should come as no surprise that the scalar field $\operatorname{div} v$ defined in the following can be interpreted as the *difference between the total inflow and the total outflow*.

Definition 1.1 (Divergence)**.** Let v be a vector field which depends on the Cartesian coordinates x, y, and z. Then the *divergence* $\operatorname{div} v$ of v is the scalar field defined by

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

We note the following important theorem:

Theorem 1.2 (Theorem 1 on page 403 in the book)**.** *The divergence* div *is independent of the choice of Cartesian coordinates.*

In the discussion above, a hole in the tube was considered a "*sink*" or a "*source*," depending on how the hole influenced the flow. The quotation marks were put there because in the full 3-dimensional description, such a sink/source behaviour would of course be accounted for in the divergence, reflecting the fact that no particles are created or annihilated, they just come from outside the tube. However, there are realistic models in which the sink/source behaviour appears in a very natural way. A simple example is a model which describes steam – vaporized water – above reservoir of liquid water. In such a model, liquid water can turn into steam and vice versa. This amounts to having *sinks* and *sources* – this time without quotation marks.

The next example explores the situation where there are no sinks or sources, but the described fluid is compressible.

Example 1.3 (Sort of Example 2 on page 404 in the book)**.** We have already established that the divergence div measures the difference between total inflow and total outflow. Let v be the velocity field of some compressible fluid (e.g. air) and ρ its density, both quantities functions of space and time. Then ρv – the density times the velocity – measures how much fluid moves in which direction pr. unit time at a given point in time and space. If we have no sinks or sources, then we have *conservation of mass*. This means that the only way that the difference between total inflow and total outflow can be different from zero is if the density changes. In fact, we have

$$\operatorname{div}(\rho v) - \frac{\partial \rho}{\partial t} = 0. \tag{1}$$

This is equation is called the *condition for conservation of mass*. In a *steady* flow, i.e. a flow which is independent of time, then (1) reduces to

$$\operatorname{div}(\rho v) = 0.$$

If the fluid is incompressible (like e.g. water and many other liquids) then (1) becomes

$$\operatorname{div}(v) = 0.$$

This is called the *condition of incompressibility*.

As a last thing before turning our attention to the curl, we note that the divergence of the gradient of a scalar field f is the Laplacian:

$$\operatorname{div}(\operatorname{grad}(f)) = \nabla^2 f, \tag{2}$$

where grad is given by

$$\operatorname{grad} v = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix},$$

and the Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The identity (2) easily follows by direct computation.

2 Curl of a vector field

[Section 9.9 in the book, p. 406]

2.1 Definition and basic properties of the curl

Again we begin with a little motivational discussion. Imagine you have some object rotating at constant angular speed $\omega > 0$ in three dimensions. Pick righthanded Cartesian coordinates so that z is the rotational axis (the axis around which the object rotates) and such that if you look in the positive direction of the z-axis, then the rotation looks clockwise – x- and y-axes must then be perpendicular to the z-axis and each other such that x, y, and z form a righthanded coordinate system.

Since the rotation is constant, it can be described by a constant (in time – not space!) vector velocity field. If we focus on a part of the object which is far away from the z-axis, clearly its speed must be large (think of wind turbines; the larger the wings, the larger the tip speed). Let us derive the velocity field. Pick any point p_0 in space, and imagine the object covers that point at time 0. By choosing r and δ correctly, we may write the coordinates of this point as

$$p_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} r\cos(\delta) \\ r\sin(\delta) \\ z \end{pmatrix}.$$

We may now parametrize the orbit of the point of the object as time goes in the following way:

$$p(t) = \begin{pmatrix} r\cos(\omega t + \delta) \\ r\sin(\omega t + \delta) \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},$$

where t is time and ω is the angular speed of the rotation (here the choice of having $\sin$ in the first coordinate and $\cos$ in the second coordinate is essential for getting the direction of the rotation right). Since p describes the *position* of the point on the object at any time, $\frac{\mathrm{d}p}{\mathrm{d}t}$ must be the *velocity field*:

$$v_p(t) = \frac{\mathrm{d}p}{\mathrm{d}t} = \begin{pmatrix} -\omega r \sin(\omega t + \delta) \\ \omega r \cos(\omega t + \delta) \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega y(t) \\ \omega x(t) \\ 0 \end{pmatrix}.$$

Evaluating this vector field at 0 gives us the velocity vector at the initial point at time 0, but as the velocity is assumed constant, this is in fact the value of the constant velocity at that given point:

$$v_p(0) = v_{p_0} = \begin{pmatrix} -\omega r \sin(\delta) \\ \omega r \cos(\delta) \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega y_0 \\ \omega x_0 \\ 0 \end{pmatrix}.$$

As p_0 was arbitrary, we now know how the velocity field looks in general and we may drop the 0's:

$$v(p) = \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix}, \qquad p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Note that all this was given by three basic facts: the angular speed, the direction of the rotation, and the axis of rotation. Now, the rotational axis is encoded in our choice of z-axis, $\omega > 0$ encodes the angular speed, and the direction could actually very easily be included by just letting ω take negative values as well. This means that all this could be encoded by a vector whose first and second entries are 0 (reflecting no rotation around the corresponding axes) and ω in the third entry:

$$w = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}.$$

Note that this same vector appears if we do the following:

$$\nabla \times v(p) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1(p) & v_2(p) & v_3(p) \end{vmatrix} = \begin{pmatrix} \frac{\partial v_3(p)}{\partial y} - \frac{\partial v_2(p)}{\partial z} \\ \frac{\partial v_1(p)}{\partial z} - \frac{\partial v_3(p)}{\partial x} \\ \frac{\partial v_2(p)}{\partial x} - \frac{\partial v_1(p)}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\omega \end{pmatrix}$$

and divide by 2. Here

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v(p) = \begin{pmatrix} v_1(p) \\ v_2(p) \\ v_3(p) \end{pmatrix},$$

and $|A|$ denotes the determinant of the matrix A. Of course we could have done with only the last entry if we just wanted to reproduce (2 times) w in our special case. The point is obviously that the formula above turns out to *always* work, independently of what the axis of rotation is, and will always point in the direction of the axis of rotation with the length being (twice) the angular speed of the rotation. In fact, it works independently of the choice of coordinates, as long as they are righthanded and Cartesian. This fact leads to the following definition:

Definition 2.1 (Curl)**.** Let $v = \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right)$ be a differentiable vector function of the Cartesian coordinates x, y, and z. Then

$$\operatorname{curl} v = \operatorname{rot} v = \nabla \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1(p) & v_2(p) & v_3(p) \end{vmatrix} = \begin{pmatrix} \frac{\partial v_3(p)}{\partial y} - \frac{\partial v_2(p)}{\partial z} \\ \frac{\partial v_1(p)}{\partial z} - \frac{\partial v_3(p)}{\partial x} \\ \frac{\partial v_2(p)}{\partial x} - \frac{\partial v_1(p)}{\partial y} \end{pmatrix}$$

is called the *curl*, the *rotor*, or the *rotation* of the vector field v.

The discussion above the definition yields the following theorem:

Theorem 2.2 (Theorem 1 on page 407 in the book)**.** *The curl of the velocity field of a rotating rigid body has the direction of the axis of rotation and its magnitude is twice the angular speed of the rotation.*

As we have already claimed, the following theorem also holds.

Theorem 2.3 (Theorem 3 on page 408 in the book)**.** *The curl of a vector field does not depend on the choice of righthanded, Cartesian coordinates.*

The interpretation of the curl as the degree of rotation at a given point leads to the following definition:

Definition 2.4 (Irrotationality)**.** A vector field v is called *irrotational* if its curl $\operatorname{curl} v$ is identically 0.

Direct computations now gives the following theorem:

Theorem 2.5. *Gradient fields are irrotational, i.e. if f is has continuous second partial derivates, then*

$$\operatorname{curl}(\operatorname{grad}(f)) = 0.$$

Moreover, if v is a vector field with continuous second partial derivatives, then the divergence of the curl is 0:

$$\operatorname{div}(\operatorname{curl}(v)) = 0.$$