Matematisk modellering og numeriske metoder

Lecture 8

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1 Fourier series

[Section 11.1 in the book, p. 474]

1.1 Periodic functions

Definition 1.1 (Periodic functions)**.** *A periodic function* f is a function on $\mathbb{R}$ with a *period* p, i.e. a number $p > 0$ such that

$$f(x) = f(x + p)$$

for all x in the domain of f, which should constitute almost all real numbers.

If $f(x) = f(x + p)$ then we also have $f(x) = f(x + p) = f((x + p) + p) = f(x + 2p)$ and, by induction, we deduce that

$$f(x) = f(x + np) \qquad \text{for all} \quad n \in \mathbb{Z}, \tag{1}$$

where $\mathbb{Z}$ is the set of integers. In particular, if f has the period p, then it is also periodic with period np for all $n \in \mathbb{N}$. The smallest positive number p such that p is a period of f is called the *fundamental period* of f. We note that if two functions f and g are both periodic with period p, then also $af + bg$ is periodic with period p, for all choices of a and b. That is, the set of functions with period p is closed under linear combinations (and hence is a vector space).

The last statement in the definition of periodic functions, namely that the domain of f should constitute *almost all* real numbers, is in fact (in spite of the loosely sounding formulation) something which has a very specific mathematical meaning, which requires a lot of theory to introduce. For our purposes, it is sufficient to note that if the exceptional set, i.e. the set of real numbers which are not in the domain of f, is *countable*, then the domain constitue almost all reals. In particular, if f has period p, then if f is undefined in a finite (or even countable) number of points in the interval $[0, p]$, then it is defined almost everywhere (i.e. for almost all $x \in \mathbb{R}$).

We already know some periodic functions, in particular $\cos$, $\sin$, $\tan$, and $\cot$. Here $\tan$ and $\cot$ are examples of functions defined almost everywhere but not everywhere: $\tan$ is undefined in $n\pi + \frac{\pi}{2}$, $n \in \mathbb{Z}$, and $\cot$ is undefined in $n\pi$, $n \in \mathbb{Z}$. They are all functions of period 2π (though two of them have a smaller *fundamental period* – which? And what are their fundamental periods?). Note, however, that they can be considered building blocks of functions with any period, since if f has period p, then for $a > 0$, f_a given by

$$f_a(x) = f(\tfrac{x}{a})$$

has period ap:

$$f_a(x + ap) = f(\tfrac{x+ap}{a}) = f(\tfrac{x}{a} + p) = f(\tfrac{x}{a}) = f_a(x).$$

This means that we may well concentrate on 2π-periodic functions in what follows, as everything easily translates to any other period. It also shows that the functions

$$x \mapsto \sin(nx) \qquad \text{and} \qquad x \mapsto \cos(nx), \qquad \text{where} \qquad n \in \mathbb{N},$$

have periods $\frac{2\pi}{n}$, and in particular, by (1), they have the period 2π. Hence linear combinations of functions of the form $\cos(nx)$ and $\sin(nx)$ are also of period 2π. A final function we will be needing, also with period 2π (to name one) is the constant function $f \equiv 1$.

Definition 1.2 (Trigonometric system)**.** The functions

$$1, \qquad \cos(x), \qquad \sin(x), \qquad \cos(2x), \qquad \sin(2x), \qquad \cos(3x), \qquad \ldots$$

form the *trigonometric system*.

1.2 Orthogonality of the trigonometric system

We will soon see that we are able to write (more or less) all "natural" 2π-periodic functions as an "infinite linear combination" of the trigonometric system. Such an "infinite linear combination" is called a *trigonometric series* and is written

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots. \tag{2}$$

Before doing that, we note that we (obviously) can write any element of the trigonometric system as such a series (by picking all *coefficients* $a_n = b_n = 0$ except for the coefficient in front of the wanted element, which should be 1).

Think of the trigonometric system as an orthogonal basis of a vector space. If we are in $\mathbb{R}^n$ and the vectors v_i, $i = 1, \ldots, n$ constitute an orthogonal basis, then any vector v can be written as a linear combination of the v_i:

$$v = \sum_{i=1}^{n} a_i v_i, \qquad \text{where} \qquad a_i = \frac{v \cdot v_i}{\|v_i\|^2}$$

In particular, if we plug in $v = v_j$, we get

$$v = v_j = \sum_{i=1}^{n} \frac{v_j \cdot v_i}{\|v_i\|^2} v_i = 0 + \cdots + 0 + \frac{v_j \cdot v_j}{\|v_j\|^2} v_j + 0 + \cdots + 0 = \frac{\|v_j\|^2}{\|v_j\|^2} v_j = v_j. \tag{3}$$

To get something similar in our case, we need something to replace the dot product, which has similar properties. Most importantly, we should be able to reproduce the coefficients $a_n = b_n = 0$ except for the coefficient 1 in front some specific function in the trigonometric system, in analogy with (3).

It turns out that the correct replacement for the dot product is the following[1]:

$$(f, g) \mapsto \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x.$$

Since $\|v\|^2 = v \cdot v$, and $\int_{-\pi}^{\pi} 1^2\,\mathrm{d}x = 2\pi$, $\int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x = \pi$, and $\int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x = \pi$, our coefficients are now:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \tag{4a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \qquad \text{and} \tag{4b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x. \tag{4c}$$

With this definition, we claim that if f is in the trigonometric system, then $a_n = b_n = 0$ for all n except for the coefficient 1 corresponding to f itself, in complete analogy with (3).

Indeed, if f is in the trigonometric system, then a_0 given by (4a) is non-zero if and only if $f \equiv 1$, in which case it is 1, and likewise for a_n and b_n: $a_n = 1$ if $f(x) = \cos(nx)$, $a_n = 0$ otherwise, and $b_n = 1$ if $f(x) = \sin(nx)$, $b_n = 0$ otherwise. (Check for yourself by plugging in $f(x) = 1$, $f(x) = \cos(mx)$ and $f(x) = \sin(mx)$ in the three expressions and see what happens if $m = n$ and if $m \neq n$).

It was perhaps no surprise that f could be written as a trigonometric series in the case where f was in the trigonometric system or that there were easy formulas for finding the coefficients in that case. We will now try to do the same thing with a somewhat different function than the ones found in the trigonometric system.

1.3 A concrete example

We now let f be given by

$$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}, \qquad f(x) = f(x + 2\pi).$$

Note that this defines f as a function almost everywhere on $\mathbb{R}$: first, we are given the values of f on $[-\pi, \pi]$ (except in the finite set $\{-\pi, 0, \pi\}$), and then we are told that f is periodic with period 2π, exactly the length of $[-\pi, \pi]$, so for any x (which cannot be written as $n\pi$ for some integer $n \in \mathbb{Z}$), we can find an $m \in \mathbb{Z}$ such that $x + m2\pi \in [-\pi, \pi]$ and then $f(x)$ must equal $f(x + m2\pi)$ because of the periodicity.

[1] This is only true for real-valued functions.

Clearly, f is not a member of the trigonometric system: it is discontinuous at $n\pi$ for all $n \in \mathbb{Z}$, while all the members of the trigonometric system are continuous. We will now see what happens if we try to write f as a trigonometric series using the coefficients given by the procedure in (4).

First, we find a_0:

$$a_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-k)\,\mathrm{d}x + \int_{0}^{\pi} k\,\mathrm{d}x\right) = 0$$

Next, we find a_n:

$$\begin{aligned}
a_n(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-k)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} k\cos(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(-k\frac{\sin(nx)}{n}\bigg|_{x=-\pi}^{0} + \frac{\sin(nx)}{n}\bigg|_{x=0}^{\pi}\right) = 0.
\end{aligned}$$

And finally, we find b_n:

$$\begin{aligned}
b_n(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int -\pi^0(-k)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} k\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(k\frac{\cos(nx)}{n}\bigg|_{x=-\pi}^{0} - \frac{\cos(nx)}{n}\bigg|_{x=0}^{\pi}\right) \\
&= \frac{k}{n\pi}(\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)) \\
&= \frac{2k}{n\pi}(1 - \cos(n\pi)) \\
&= \frac{2k}{n\pi}(1 - (-1)^n),
\end{aligned}$$

so $b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$, $b_4 = 0$, $b_5 = \frac{4k}{5\pi}$, and so on. This means that the Fourier series is

$$\frac{4k}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right).$$

This series in fact converges to f. What this means is not made precise in the book, and the strongest sense in which the series converges is far beyond the scope of this course. For now, we just note that we have pointwise convergence on the domain of f, i.e. for each x such that $f(x)$ is defined,

$$f(x) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

In particular, $f(\frac{\pi}{2}) = k = \frac{4k}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots)$, implying that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots,$$

which is a non-trivial result. We now turn to the question of when the Fourier series actually converges.

1.4 Convergence of the Fourier series

Before stating the main theorem of this section, which gives sufficient conditions for pointwise convergence valid in most practical applications, we need to get some new notions defined.

Definition 1.3 (left-hand limit, right-hand limit)**.** Let f be a function which is defined in a neighborhood to the left of x_0. If the limit

$$f(x_0 - 0) = \lim_{h \uparrow 0} f(x_0 + h)$$

exists, where $h \uparrow 0$ means that the limit is taken through negative numbers ("to zero from below"), then $f(x_0 - 0)$ is called the *left-hand limit* of f at x_0. Likewise, if f is defined in a neighborhood to the right of x_0, then if

$$f(x_0 + 0) = \lim_{h \downarrow 0} f(x_0 + h)$$

exists, where $h \downarrow 0$ means that the limit is taken through positive numbers, then $f(x_0 + 0)$ is called the *right-hand limit* of f at x_0.

Of course, the notation $f(x_0 + 0)$ is somewhat unfortunate, as it already has a different meaning. However, usually, the meaning can easily be deduced from the circumstances. Having now established left-hand and right-hand limits, we are fully equipped for the next definition.

Definition 1.4 (left-hand derivative, right-hand derivative)**.** The left-hand derivative of f at x_0 (if it exists) is given by

$$\lim_{h \uparrow 0} \frac{f(x_0 + h) - f(x_0 - 0)}{h}.$$

The right-hand derivative of f at x_0 is correspondingly

$$\lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$

when it exists.

A few remarks: There is a mistake in the book (a sign error) in the definition of right-hand derivatives. If f is continuous at x_0, then the left- and right-hand limits $f(x_0 - 0), f(x_0 + 0)$ are both just $f(x_0)$. If f is differentiable at x_0, then the left- and right-hand derivatives are equal to each other and to the ordinary derivative.

We are now ready to state the theorem.

Theorem 1.5. *Let f be 2π-periodic and piecewise continuous with left- and right-hand derivatives everywhere. Then the Fourier series* (2) *with the coefficients given by* (4) *converges pointwise to f except where f is discontinuous. At the discontinuity points of f, the Fourier series converges to the left- and right-hand limits of f at this point.*

To illustrate the concepts and the theorem, we consider our previous example,

$$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}, \qquad f(x) = f(x + 2\pi).$$

This function is, as already noted, 2π-periodic. It is also piecewise continuous, it has left- and right-hand derivatives everywhere (what are their values at $-\pi$, 0, and π?), so if we believe the theorem above, its Fourier series converges to f pointwise except at $n\pi$, $n \in \mathbb{Z}$, at which the series converges to the average of the left- and right-hand limits of f.

The left-hand limit of f at $2n\pi$ is $-k$, while the left-hand limit of f at $(2n+1)\pi$ is k, $n \in \mathbb{Z}$. The right-hand limit of f at $2n\pi$ is k, while the right-hand limit of f at $(2n+1)\pi$ is $-k$. This means that whatever n is, the average of the left- and right-hand limit at $n\pi$ is always a big fat 0. Does this agree with what we know about the Fourier series by just plugging in $x = n\pi$?