Matematisk modellering og numeriske metoder

Lecture 9

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1 Fourier series

[Section 11.2 in the book, p. 483]

1.1 Changing periods in connection with Fourier series

We already saw last time, that if f has period p, then f_a given by

$$f_a(x) = f(\tfrac{x}{a})$$

has period ap. This trick of course also works in connection with Fourier series. In particular, if we are interested in a period of, say, $2L$, the Fourier setup can be translated in the following way:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \end{aligned}$$

for $n = 1, 2, 3, \ldots$. There are several detailed examples on the use of this setup in the book on the pages 484–486.

1.2 Simplifications for even and odd functions

Last time, we noted that the odd function f given by

$$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$$

had a Fourier series consisting only of $\sin$ terms. We also noted that f and all the $\sin$ terms are odd functions, while the a_0 and $\cos$ terms are even functions (one can in fact consider the a_0 a $\cos$ term with $n = 0$). We recall that an odd function is a function g which satisfies

$$g(-x) = -g(x)$$

while an even function h is a function which satisfies

$$h(-x) = h(x).$$

The point is of course that an odd function integrates to 0 over $[-\pi, \pi]$, and products of two odd functions or two even functions are even, while the product of an odd and an even function is odd (just like with *sums* of even and odd numbers). On top of that, if we know a 2π-periodic function f on the interval $[0, \pi]$, and we know that it is even (or odd), then we know it everywhere. Because of this, it is also enough to integrate over half of the interval when finding Fourier coefficients of even or odd functions. Put more precisely:

Theorem 1.1 (Summary on page 487 in the book)**.** *Let f be a 2π-periodic function whose Fourier series converges pointwise to f. If f is even (i.e. $f(-x) = f(x)$), then the Fourier series reduces to*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \qquad \text{and} \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad \text{for } n = 1, 2, 3, \dots .$$

If f is odd (i.e. $f(-x) = -f(x)$), then the Fourier series reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

Since all functions f can be written as a sum of an odd function f_1 and an even function f_2, the above theorem can be used in connection with the following theorem, whose proof follows easily from the linearity of integrals.

Theorem 1.2 (Theorem 1 in the book on page 487)**.** *If we write $a_0(f)$, $a_n(f)$, and $b_n(f)$ for the Fourier coefficients of a function f, then the* functionals *a_0, a_n, and b_n are* linear, *i.e.*

$$a_0(f_1+f_2) = a_0(f_1)+a_0(f_2), \quad a_n(f_1+f_2) = a_n(f_1)+a_n(f_2), \quad \text{and} \quad b_n(f_1+f_2) = b_n(f_1)+b_n(f_2)$$

and

$$a_0(cf) = ca_0(f), \qquad a_n(cf) = ca_n(f), \qquad \text{and} \qquad b_n(cf) = cb_n(f),$$

for any functions f_1, f_2, and f and any real number c. In words, the Fourier coefficients of a sum f_1+f_2 is a sum of the Fourier coefficients of f_1 and f_2, and the Fourier coefficients of cf is c times the Fourier coefficients of f.

The above theorem also proves useful in situations where the function f one wants to find the Fourier expansion of is naturally written as a sum.

1.3 Half range expansions

The reduced complexity of odd and even Fourier expansions and their simpler calculations (you only need to integrate half of the interval) inspires the following idea. Imagine you have a function which is naturally defined on the finite interval $[0, L]$ but you would like to express it as a Fourier series (i.e. as a periodic function). Naturally, one could say "well, I just say that my function is L-periodic and use the trick of Section 1.1 of the present note!" which is of course completely legal, especially of one notes that the formulas written in that section pertains to functions of period $2L$, not L, and one then adjusts accordingly. However, if we are not really interested in the periodicity of the function, just on the Fourier expansion, then one may *consider* the function to be $2L$ periodic *and even* (or odd, depending on what gives the simplest Fourier expansion). Put a bit more schematically, we get:

1. Denote by f a function on $[0, L]$ we want to get a Fourier expansion of.
2. Let f_1 be an *even, $2L$-periodic extension* of f, and likewise, let f_2 be an *odd, $2L$-periodic extension* of f.
3. Pick whichever seems more pratical of f_1 and f_2 and apply Theorem 1.1 to this function, after suitable adjustments of the formulas to the $2L$ case in the spirit of Section 1.1 (i.e. π's should be L's and n's inside trigonometric functions should be multiplied by $\frac{\pi}{L}$).
4. The result is a $2L$-periodic function which agrees with f on $[0, L]$ as long as f is sufficiently nice (see Theorem 1.5 from the Lecture 8 Notes for sufficient conditions for being sufficiently nice).

To be more concrete, assume that f_2 is the more practical choice. Then the Fourier coefficients of f_2 are $a_0(f_2) = a_n(f_2) = 0$ for $n = 1, 2, 3, \ldots$ and

$$b_n(f_2) = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x.$$