Problem 1 (40 POINT)

In this problem we assume that $-1 < x < 1$.

(a) Rewrite the ODE

$$\frac{y'(x)}{x} - 2y^2(x) = 0 \tag{1}$$

to separated form.

(b) Solve the ODE (1).

(c) Solve the intial value problem given by (1) and $y(0) = 1$ and call the solution y_1.

(d) Show that the solution y_1 from (c) is also a solution to

$$y''(x) + \frac{2x^3 + 6x}{x^4 - 1} y'(x) = -\frac{2}{x^4 - 1}. \tag{2}$$

(e) Show that $y_2(x) = \frac{1}{1-x}$ is also a solution to (2).

(f) Note that $y_1(0) = y_2(0) = 1$. Can you conclude that $y_2 = y_1$? Explain.

(g) Find a solution y_3 to

$$y''(x) + \frac{2x^3 + 6x}{x^4 - 1} y'(x) = 0. \tag{3}$$

If possible, you may use y_1 and/or y_2 in the expression of y_3, even if you haven't solved (c).

(h) Find a solution y_4 to (3), which is linearly independent of y_3. It is enough to find a formula for the solution, you are allowed to use y_1, y_2, and/or y_3 in the expression udtrykket, even if you haven't found these, and you are not required to compute possible integrals.

(i) Find the general solution y_h to (3). You are allowed to use y_1, y_2, y_3, and/or y_4 in the expression, even if you haven't found these.

(j) Find the general solution y_g to (2). You are allowed to use y_1, y_2, y_3, y_4, and/or y_h in the expression, even if you haven't found these.

(k) Find a solution y_p to (2) satisfying $y_p(0) = 1$ and $y_p'(0) = 1$.

(l) Kan you find other solutions $\tilde{y}_p$ to (2) which satisfy $\tilde{y}_p(0) = 1$ and $\tilde{y}_p'(0) = 1$? Explain.

Problem 2 (20 POINT)

In this problem we solve the intial value problem

$$y'''(t) - \frac{107}{27}y(t) = 198e^{-\frac{t}{3}}\sin(2t), \quad y(0) = 27, \quad y'(0) = -9, \quad y''(0) = -105$$

using the Laplace transform.

(a) Find the Laplace transforms $F = \mathcal{L}(f)$ and $G = \mathcal{L}(g)$ of $f(t) = e^{-\frac{t}{3}}$ and $g(t) = \sin(2t)$, resp.

(b) Let $H = \mathcal{L}(h)$ denote the Laplace transform of $h(t) = e^{-\frac{t}{3}}\sin(2t)$. Can one compute H by computing the product FG?

(c) Compute the Laplace transform $I = \mathcal{L}(i)$ of $i(t) = 198e^{-\frac{t}{3}}\sin(2t)$.

(d) Find an equation for $Y = \mathcal{L}(y)$, where y is the solution to the intial value problem above and isolate Y in the expression. Hint: remember that $\mathcal{L}(y''')(s) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$.

(e) Find the inverse Laplace tranform $\mathcal{L}^{-1}(J)$ of $J(s) = \frac{s+\frac{1}{3}}{(s+\frac{1}{3})^2+2^2}$.

(f) Find the solution to the intial value problem.

Problem 3 (20 POINT)

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n!}$, which can also be written $f(x) = e^{\cos(x)}\sin(\sin(x))$.[1]

(a) Is f periodic? If yes, find the fundamental period of f.

(b) Is f even, odd, both, or neither? Explain.

(c) Let $u(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!}\cos(2nt)\sin(nx)$. Show that u solves the one-dimensional wave equation $u_{tt} = c^2u_{xx}$ on $(x, t) \in [0, \pi] \times \mathbb{R}_{\geq 0}$ with boundary condition $u(0, t) = u(\pi, t) = 0$ for a suitable $c > 0$ and determine c.

(d) Find g such that u satisfies the intial value condition $u_t(x, 0) = g(x)$.

(e) Find h such that u satisfies the intial value condition $u(x, 0) = h(x)$.

(f) Find the Fourier series of h.

[1] You don't have to worry about why $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n!} = e^{\cos(x)}\sin(\sin(x))$.

Problem 4 (20 POINT)

Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n!}$ as in Problem 3.

(a) One can show that $\sum_{n=0}^{\infty} \frac{1}{n!} = e \approx 2.71828$. Argue that $\left|\sum_{n=5}^{\infty} \frac{\sin(nx)}{n!}\right| < 0.01$, independently of what x is. Hint: Look at $\left|e - \sum_{n=0}^{4} \frac{1}{n!}\right|$ and remember that $0! = 1$, that $\sin(0 \cdot x) = 0$, and that $|\sin(x)| \leq 1$.

(b) Let $f_4(x) = \sum_{n=1}^{4} \frac{\sin(nx)}{n!}$. According to (a), one only makes a small mistake by using f_4 instead of f. Use this and the identities $\sin(\frac{\pi}{4}) = \sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$, $\sin(\frac{\pi}{2}) = 1$, and $\sin(\pi) = 0$ to – without using a calculator – find a sum of (at most[2]) four numbers that approximate $f(\frac{\pi}{4})$ with an error less than 0.01.

(c) It turns out that the sum from (b) is *larger* than $f(\frac{\pi}{4})$, about 0.0074 too big. Explain why $f_4(x)$ can get larger than $f(x)$ for some values of x.

From now on, we will focus on $f_2(x) = \sum_{n=1}^{2} \frac{\sin(nx)}{n!} = \sin(x) + \frac{\sin(2x)}{2}$, for simplicity, even though it means that we risk errors with an absolute value as large as $e - 2.5 \approx 0.22$.

(d) Let p_4 denote the fifth order polynomial that agrees with f at $x = 0$, $x = \frac{\pi}{4}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{4}$, and $x = \pi$, and let q_4 denote a fifth order polynomial that agrees with f_2 at the same points. Remember that $f(x) = e^{\cos(x)} \sin(\sin(x))$. Which values of $\sin$, $\cos$, and $\exp$ do we need to know to compute p_4, and which values of $\sin$ do we need to know to compute q_4? The answer may be given as a table of values.

The above questions deals with approximating f in diffferent ways.

(e) One can show that f is a solution to the differential equation $y'(x) = e^{\cos(x)} \cos(x + \sin(x))$, and it is easy to see that $f(0) = 0$. Explain how this information can also be used to approximate f.

(f) Estimate $\int_0^1 f(x)\,\mathrm{d}x$ using Simpson's rule with $n = 1$ subintervals. It is allowed to use a computer and/or calculator.

[2]The sum can easily be reduced to just two terms