Problem 1 (20 POINTS)

In this problem, we assume that $0 < x < \frac{\pi}{2}$.

(a) Write the ODE

$$\tan(x)y'(x) = y(x) \tag{1}$$

in separated form.

(b) Use separation of variables to find an equation for a solution y to (1) which doesn't contain y'. You are not required to compute the integrals.

(c) Let y_1 denote the solution to (1) which satisfies r $y(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. Find $y_1'(\frac{\pi}{4})$.

(d) The solution y_1 is also a solution to

$$y''(x) + \tan(x)y'(x) + y(x) = \sin(x). \tag{2}$$

Use (1) and (2) to find a different first order linear ODE for which y_1 is a solution. Hint: isolate y' in (1) and find an expression for y'' by differentiating on both sides of the equation.

(e) Find $y_1''(\frac{\pi}{4})$.

(f) Find y_1. Explain how you did.

Problem 2 (20 POINTS)

Let $y_1(x) = \sin(x)$ and $y_2(x) = \sin(x)(\cos(x) + 1)$. One can show that y_1 and y_2 are solutions to

$$y''(x) + \sin(x)\cos(x)y'(x) + (4 - \cos(2x))y(x) = \frac{15}{4}\sin(x) - \frac{1}{4}\sin(3x). \tag{3}$$

(a) Find a solution $y_3 \neq 0$ to

$$y''(x) + \sin(x)\cos(x)y'(x) + (4 - \cos(2x))y(x) = 0. \tag{4}$$

(b) Find a solution y_4 to (4) which is linearly independent of y_3. It's enough to find an expression for the solution, you are allowed to use y_1, y_2, and/or y_3 in the expression, even if you haven't found these, and if the expression contains integrals, you are not required to compute them.

(c) Find the general solution y_h to the homogeneous ODE (4). You are allowed to use y_1, y_2, y_3, and/or y_4 in the expression, even if you haven't found these.

(d) Find the general solution y_g to (3). You are allowed to use y_1, y_2, y_3, y_4, and/or y_h in the expression, even if you haven't found these.

(e) Find a solution y_p to (3) satisfying $y_p(0) = 0$ and $y_p'(0) = 2$.

(f) Can you find other solutions $\tilde{y}_p$ to (3), which satisfy $\tilde{y}_p(0) = 0$? Explain.

Problem 3 (20 POINTS)

In this problem the following initial value problem should be solved:

$$y''(t) + \tfrac{444}{23}y'(t) + \tfrac{51400}{529}y(t) = -10e^{-\frac{222}{23}t}\sin(3t), \tag{5}$$

$$y(0) = 0, \quad y'(0) = 6. \tag{6}$$

(a) Find the general solution y_h to the homogeneous problem

$$y''(t) + \tfrac{444}{23}y'(t) + \tfrac{51400}{529}y(t) = 0. \tag{7}$$

(b) Find a solution y_1 to (7) satisfying $y_1(0) = 0$ and $y_1'(0) = 2$.

(c) Is it also true that the function $y_2(t) = 2e^{-\frac{222}{23}t}\sin(t)\cos(t)$ is a solution to (7) with $y_2(0) = 0$ and $y_2'(0) = 2$? Explain.

(d) Is $y_1 = y_2$? Explain.

(e) Use the method of undetermined coefficients to find a particular solution y_p to (5).

(f) Find the solution to the initial value problem given by (5) and (6).

Problem 4 (20 POINTS)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}$ which can also be written $f(x) = e^{\cos(x)}\cos(\sin(x))$.[1]

(a) Is f periodic? If yes, find the fundamental period for f, i.e. the shortest positive period of f.

(b) Is f even, odd, both or neither? Explain.

(c) Let $u(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\cos(nx)e^{-4n^2t}$. Show that u solves the onedimensional heat equation $u_t = c^2u_{xx}$ on $(x,t) \in [0,\pi] \times \mathbb{R}_{\geq 0}$ for a suitable $c > 0$ and determine the correct value of c.

(d) Which boundary condition does $u(x,t)$ satisfy?

(e) Find h, such taht u satisfies the initial value condition $u(x,0) = h(x)$.

(f) Find the Fourier series for h.

[1] You don't have to worry about why $\sum_{n=0}^{\infty} \frac{\cos(nx)}{n!} = e^{\cos(x)}\cos(\sin(x))$.

Problem 5 (20 POINT)

Denote the right-hand side in (3) from Problem 2 by f, i.e. $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \frac{15}{4}\sin(x) - \frac{1}{4}\sin(3x)$.

(a) Find the Fourier series for f. Explain.

(b) Find a second order polynomial p_2 which agrees with f at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$.

(c) Is f and/or p_2 periodic? And if, what is the (fundamental) period?

(d) Use Simpson's method with $n = 2$ (two subintervals) to estimate $\int_0^\pi f(x)\,\mathrm{d}x$.

(e) Estimate the error in the previous result.

(f) Can one find a solution to $f(x) = x$ using fixed point iteration? Explain.