14.1 Project
Midpoint Sums Approximating Double Integrals

Suppose we subdivide the intervals $[a,b]$ and $[c,d]$ into m subintervals of length Δx and into n subintervals of length Δy (respectively). If

$$u_i = a + (i - \tfrac{1}{2})\Delta x \quad \text{and} \quad v_j = c + (j - \tfrac{1}{2})\Delta y \qquad (1)$$

denote the midpoints of the *i*th *x*-subinterval and the *j*th *y*-subinterval (respectively), then (u_i, v_j) is the midpoint of the *ij*th subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. We therefore get the midpoint sum approximation

$$S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(u_i, v_j)\,\Delta x\,\Delta y \approx \iint_R f(x,y)\,dA \qquad (2)$$

to the double integral of the function f over the rectangle $R = [a,b] \times [c,d]$.

In the paragraphs below we illustrate the use of computer algebra systems to calculate the approximation in (1) with the function $f(x,y) = 4x^3 + 6xy^2$ and the rectangle $R = [1,3] \times [-2,1]$ of Example 1 in the text. You can use these techniques to calculate midpoint approximations to integrals such as the ones listed below (or to integrals of your own choice). Compare your numerical approximations using various numbers m and n of subintervals with the exact values of these integrals.

1. $\int_0^1 \int_0^1 (x+y)\,dy\,dx$

2. $\int_0^3 \int_0^2 (2x+3y)\,dy\,dx$

3. $\int_0^2 \int_0^2 xy\,dy\,dx$

4. $\int_0^1 \int_0^1 x^2 y\,dy\,dx$

5. $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin y\,dy\,dx$

5. $\int_0^{\pi/2} \int_0^1 \frac{\cos x}{1+y^2}\,dy\,dx$

Using Maple

First we define the desired function $f(x,y)$, the rectangle $R = [a,b] \times [c,d]$, the desired numbers m and n of *x*-subintervals and *y*-subintervals (respectively), and the resulting subinterval lengths Δx and Δy.

```
f := (x,y)->4*x^3 + 6*x*y^2:
```

```
a:=1:  b:=3:     c:=-2:  d:=1:
m:=8:  n:=12:
dx := (b-a)/m:   dy:=(d-c)/n:
```

Then the subinterval midpoints in (1) are defined by

```
u := i:
v := j->c+(j-1/2)*dy:
```

We are now ready to calculate and evaluate numerically the midpoint sum in (2).

```
S := sum( sum(f(u(i),v(j)), j=1..n), i=1..m)*dx*dy:
evalf(S);
                    310.8750000
```

For comparison, the exact value of our integral is given by

```
int( int(f(x,y),x=a..b), y=c..d);
                        312
```

Using Mathematica

First we define the desired function $f(x, y)$, the rectangle $R = [a,b]\times[c,d]$, the desired numbers m and n of x-subintervals and y-subintervals (respectively), and the resulting subinterval lengths Δx and Δy.

```
f[x_, y_] = 4*x^3 + 6*x*y^2;
a = 1;  b = 3;    c = -2;  d = 1;
m = 8;  n = 12;
dx = (b - a)/m;   dy = (d - c)/n;
```

Then the subinterval midpoints in (1) are defined by

```
u[i_] = a + (i-1/2)*dx;
v[j_] = c + (j-1/2)*dy;
```

We are now ready to calculate and evaluate numerically the midpoint sum in (2).

```
S = Sum[Sum[f[u[i],v[j]],{j,1,n}],{i,1,m}]*dx*dy // N
310.875
```

For comparison, the exact value of our integral is given by

```
Integrate[Integrate[f[x,y], {x,a,b}], {y,c,d}]
```

```
312
```

Using MATLAB

First we define the desired function $f(x, y)$, the rectangle $R = [a,b]\times[c,d]$, the desired numbers m and n of x-subintervals and y-subintervals (respectively), and the resulting subinterval lengths Δx and Δy.

```
f = inline('4*x.^3 + 6*x.*y.^2')
a = 1;  b = 3;    c = -2;  d = 1;
m = 8;  n = 12;
dx = (b - a)/m;   dy = (d - c)/n;
```

Then vectors consisting of the subinterval midpoints in (1) are defined by

```
i = 1 : m;   j = 1 : n;
u = a + (i - 1/2)*dx;
v = c + (j - 1/2)*dy;
```

Next, and array consisting of the midpoints of the mn subrectangles is given by

```
[u,v] = meshgrid(u,v);
```

We are now ready to calculate and evaluate numerically the midpoint sum in (2).

```
sum( sum(f(u,v) ))*dx*dy
ans =
  310.8750
```

For comparison, a highly accurate value of our integral is given by

```
dblquad(f, a,b, c,d)

ans =
   312
```