Stability of Multidimensional Persistent Homology Groups

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Outline

1. **The Persistent Topology approach to shape comparison**
2. Multidimensional Persistent Homology
3. Finiteness of Rank Invariants
4. Representation of rank invariants via persistence diagrams
5. Comparison of Rank Invariants
6. Stability with respect to noisy functions
7. Stability with respect to noisy domains

What is the shape of an object?

- Oxford Dictionary: the external form[1] or appearance of someone or something as produced by their outline

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- Mathematically: no universally accepted definition
  - outline, surface
  - up to rigid motions, or affinities, or perspective transformations
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- Mathematically: no universally accepted definition
  - outline, surface
  - up to rigid motions, or affinities, or perspective transformations
  - may or may not take into account color or texture information
- Most of the proposed techniques for shape recognition are tailored for some particular interesting cases
  - polyhedral rigid objects
  - planar curves
  - point cloud data

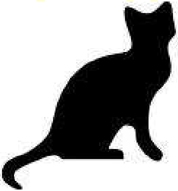
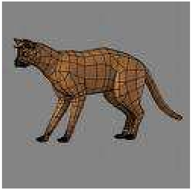
[1] Form: Visible shape or configuration

What is the shape of an object?

- Tentative definitions are generally based on *observers' perceptions*.
- Dependence on observers implies large subjectivity
  - changes due to object orientation and distance from the object
  - changes due to light conditions
- Human judgments focus on *persistent perceptions*
  - Non-persistent properties can be considered as noise.
  - Only *stable perceptions* concur to give a shape to objects.

How to model observations?

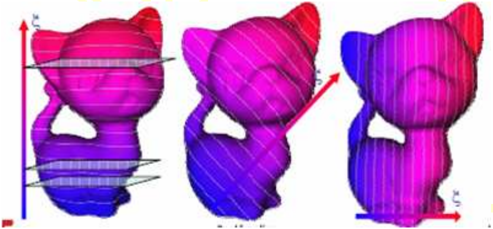
- A set of observations can be modeled as a topological space X.
- The topological space depends on what the observer is observing: boundary, interior, projection, contour

How to model perceptions?

- Observer's perceptions can be modeled as a function $f : X \to \mathbb{R}^k$.
- The function depends on the shape property the observer is perceiving: curvature, roundness, elongation, connectivity etc.
- For each observation $x \in X$, f describes x as seen by the observer.

Thus we are led to study pairs (X, f) where
– X is a topological space
– $f : X \to \mathbb{R}^k$ a (continuous) function, called a measuring (filtering) function.



Shape comparison

In general shape comparison amounts to giving a measure of dissimilarity between shapes.

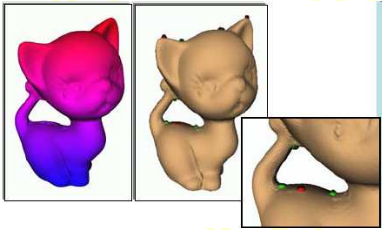
- When two pairs (X_1, f_1) and (X_2, f_2) are a comparable set of observations and perceptions, it is natural to ask how dissimilar they are.
- Persistent Topology proposes an approach where comparing shapes means comparing properties expressed by real functions

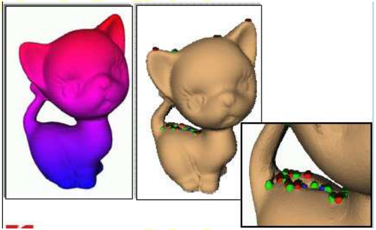
$d\Big($  $+ f_1,$  $+ f_2 \Big) =?$

About Stability (1)

In order that comparisons be reliable we need:

- Stability in the perceptions





yielding to a request for

stability w.r.t. perturbations of the function

About Stability (2)

- Stability in the observations

Buddha has genus 104,
Dragon has genus 46,
David's head has genus 340.

Most of these tunnels/handles are artifacts of the acquisition process of volumetric data (Guskov-Wood, Topological noise removal)

yielding to a request for

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Persistent Topology Tools

size functions [Frosini '91]

size homotopy groups [Frosini-Mulazzani '99]

persistent homology groups [Edelsbrunner-Letscher-Zomorodian '00]

vines and vineyards [CohenSteiner-Edelsbrunner-Morozov '06]

interval persistence [Dey-Wenger '07]

multidimensional homology groups [Carlsson-Zomorodian '07]

Main idea:

Given a space X,

- take a vector-valued function $\vec{f} : X \to \mathbb{R}^k$;
- consider the collection of nested lower level sets of $\vec{f}$;
- encode the scale at which a topological feature (e.g., a connected component, a tunnel, a void) is created, and when it is annihilated along this filtration using homology groups;
- further encode this information using a parametrized version of Betti numbers.

Formally:

Given a space X and a continuous function $\vec{f}: X \to \mathbb{R}^k$,

Lower level sets

For every $\vec{x} \in \mathbb{R}^k$, $X\langle \vec{f} \preceq \vec{u} \rangle = \{x \in X : \vec{f}(x) \preceq \vec{u}\}$.
($(u_1, \ldots, u_k) \preceq (v_1, \ldots, v_k)$ means $u_j \leq v_j$ for every index j.)

Definition (Carlsson&Zomorodian 2007)

The multidimensional persistent homology groups of $(X, \vec{f})$ are the groups

$$H_q^{\vec{u},\vec{v}}(X, \vec{f}) = ImH_q\left(X\langle \vec{f} \preceq \vec{u} \rangle \hookrightarrow X\langle \vec{f} \preceq \vec{v} \rangle\right)$$

for $\vec{u} \prec \vec{v}$.

Homology coefficients taken in a field $\mathbb{K}$

Definition (Carlsson&Zomorodian 2007)

The rank invariants of $(X,\vec{f})$ are functions

$$\rho_{(X,\vec{f}),q}:\{\vec{u}\prec\vec{v}\}\to\mathbb{N}\cup\{\infty\},\ q\in\mathbb{Z},$$

such that $\rho_{(X,\vec{f}),q}(\vec{u},\vec{v})$ equals the rank of the persistent homology group $H_q^{\vec{u},\vec{v}}(X,\vec{f})$.

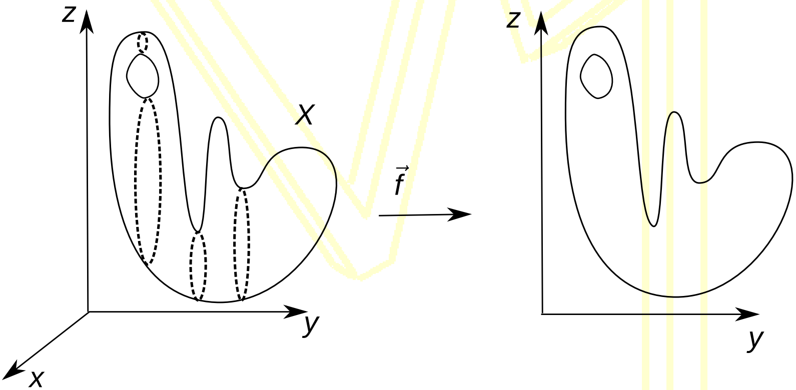
Case $q=0$:

Rank invariants are also called size functions [Frosini et al. 1991,....]

Example of Rank Invariant

$\vec{f}: X \to \mathbb{R}^2$, $\vec{f} = (y, z)$, $\rho_{(X,\vec{f}),1} : \{\vec{u} \prec \vec{v} \in \mathbb{R}^2 \times \mathbb{R}^2\} \to \mathbb{N}$

$\rho_{(X,\vec{f}),1}(\vec{u}, \vec{v}) =$ 1-homology classes born before $\vec{u}$ and still alive at $\vec{v}$



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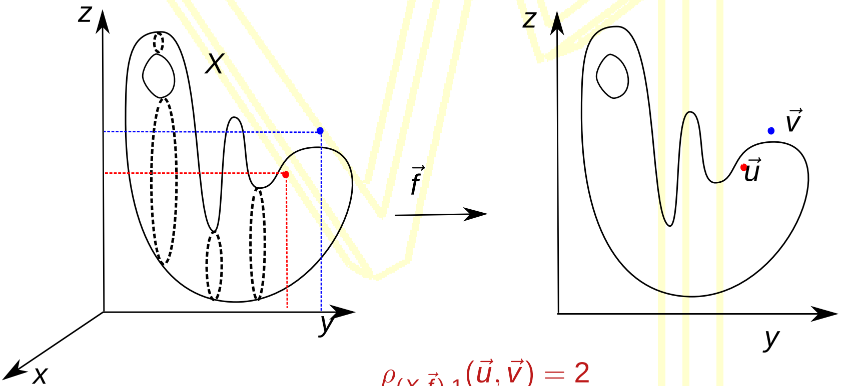

$\rho_{(X,\vec{f}),1}(\vec{u}, \vec{v}) = 2$

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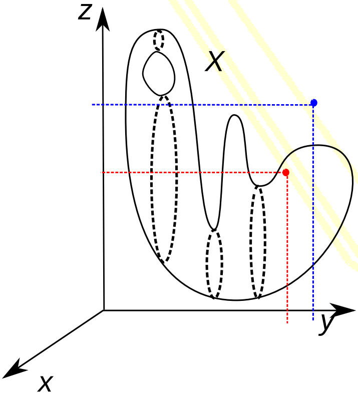

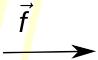

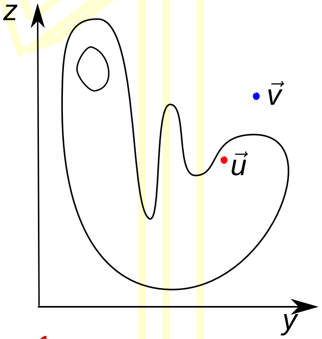

$\rho_{(X,\vec{f}),1}(\vec{u}, \vec{v}) = 1$

Example of Rank Invariant

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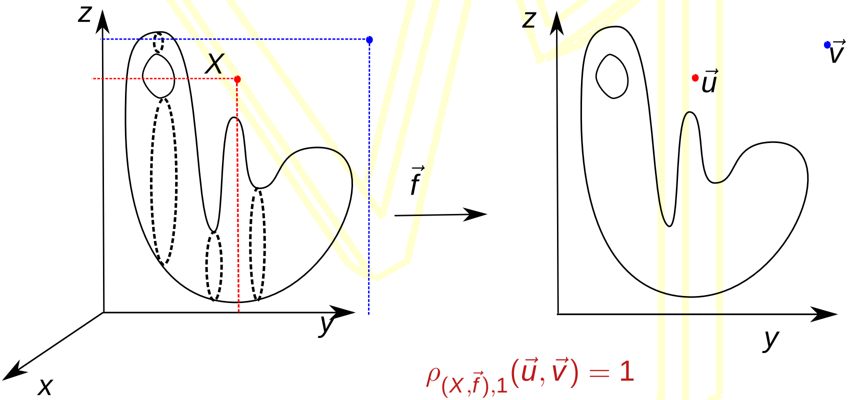


$\rho_{(X,\vec{f}),1}(\vec{u}, \vec{v}) = 1$

Issues

- Finiteness of rank invariants;
- Representation of rank invariants via persistence diagrams
- Comparison of rank invariants;
- Stability of rank invariants with respect to perturbations of $\vec{f}$;
- Stability of rank invariants with respect to perturbations of X;

Observation

Case $q = 0$ (size functions):
X compact and locally connected, $\vec{f} : X \to \mathbb{R}^k$ continuous imply

$$\rho_{(X,\vec{f}),0}(\vec{u},\vec{v}) = \mathrm{rk}\ \mathrm{Im}\ H_0 \left(X\langle \vec{f} \preceq \vec{u}\rangle \hookrightarrow X\langle \vec{f} \preceq \vec{v}\rangle \right) < +\infty$$

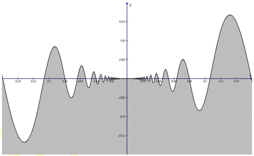
for every $\vec{u} < \vec{v}$.

Definition (Cohen-Steiner et al. 2005)

X triangulable space, $f : X \to \mathbb{R}$ is tame if it has a finite number of homological critical values and $H_q(X\langle f \le u\rangle)$ is finite-dimensional for every $q \in \mathbb{Z}$ and $u \in \mathbb{R}$.

Tame functions are not closed under the max operator:

- $f_1(u,v) = \begin{cases} v - u^2 \sin \frac{1}{u} & u \neq 0, \\ v & u = 0. \end{cases}$



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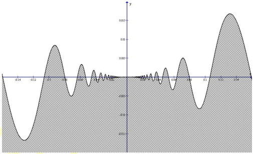
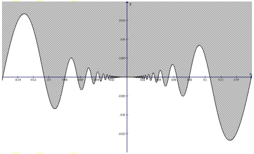


$f_1(u,v) \leq 0$

- $f_2(u,v) = \begin{cases} -v - u^2 \sin \frac{1}{u} & u \neq 0, \\ -v & u = 0. \end{cases}$



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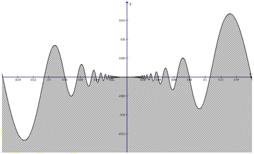
- $f(u,v) = \max\{f_1, f_2\}$



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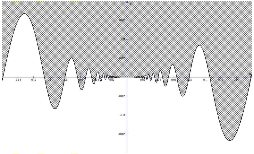

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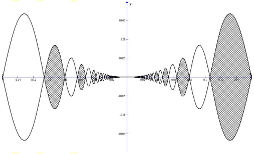


$f(u,v) \leq 0$

Definition

A topological subspace Y of $\mathbb{R}^n$ is called a Euclidean neighborhood retract if there is a neighborhood in $\mathbb{R}^n$ of which Y is a retract.

Theorem (Borsuk)

If $Y \subseteq \mathbb{R}^n$ is locally compact and locally contractible then Y is an ENR.

Theorem

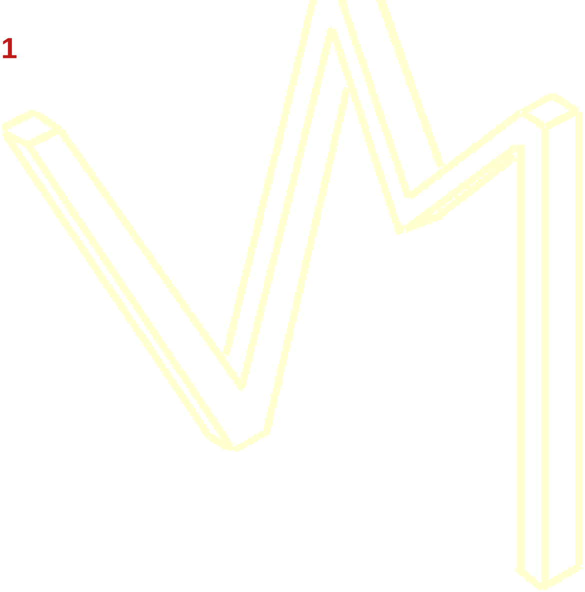
If $X \subseteq \mathbb{R}^n$ is compact and locally contractible then

$$\rho_{(X,\vec{f}),q}(\vec{u},\vec{v}) < +\infty$$

for every $\vec{u} < \vec{v}$, $q \in \mathbb{Z}$, $\vec{f} : X \to \mathbb{R}^k$ continuous.

Persistence Diagrams (1)

Case $\mathbf{k} = \mathbf{1}$



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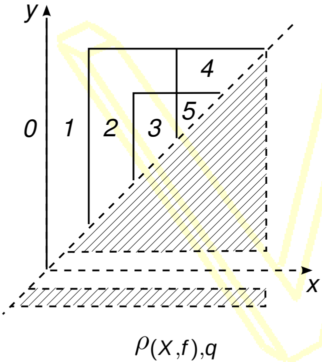

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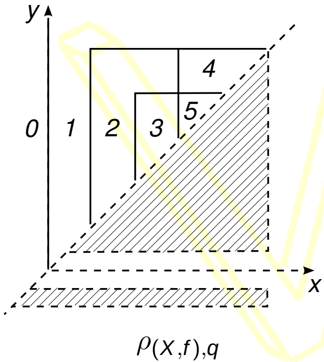

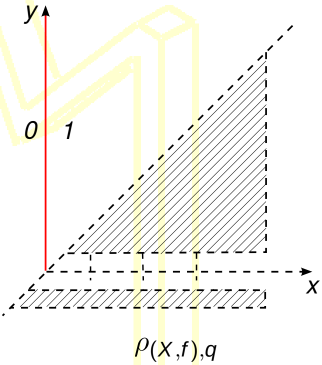

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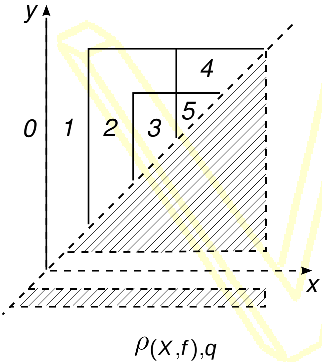

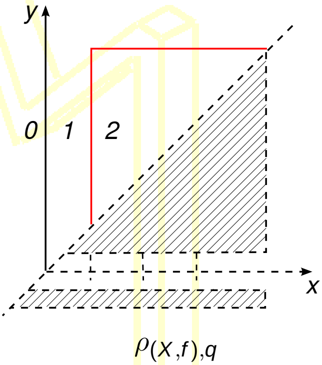

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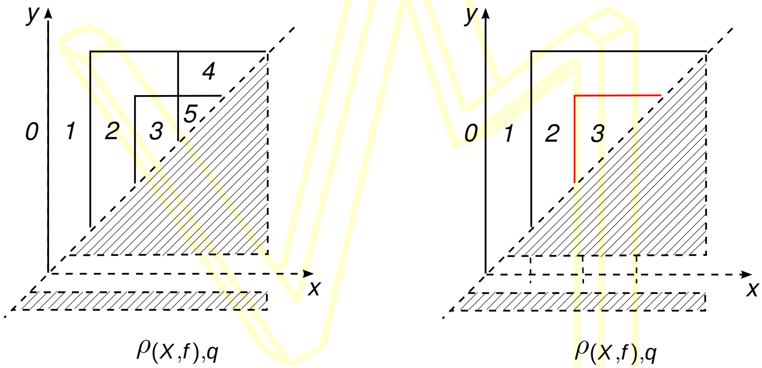

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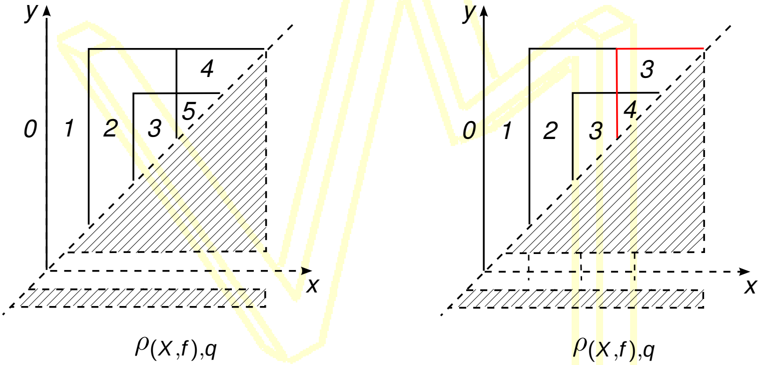

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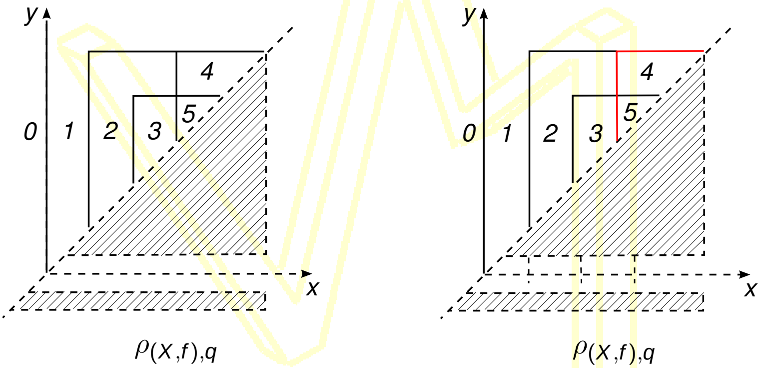

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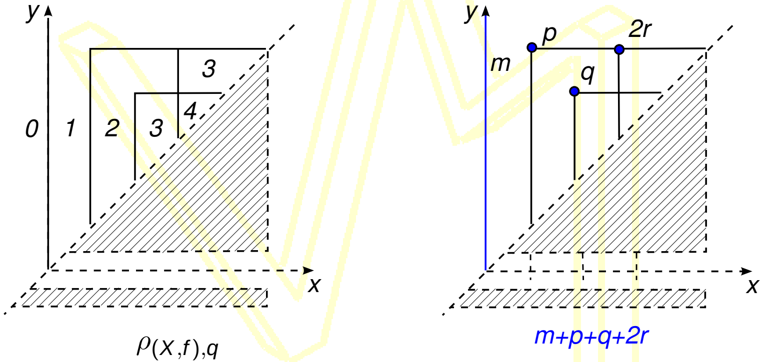


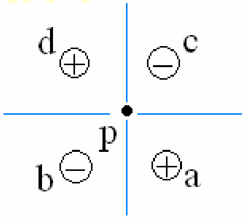
Persistence Diagrams (2)

Definition

The multiplicity of a point $p = (u, v)$ s. t. $u < v$ is the number

$$\mu(p) := \lim_{\epsilon \to 0+} \rho_{(X,f),q}(u+\epsilon, v-\epsilon) - \rho_{(X,f),q}(u+\epsilon, v+\epsilon) - \rho_{(X,f),q}(u-\epsilon, v-\epsilon) + \rho_{(X,f),q}(u-\epsilon, v+\epsilon).$$

If $\mu(p) > 0$ the point p is called a cornerpoint.



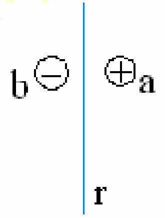
Cornerlines

Definition

A vertical line $r : u = k$ such that the number

$$\mu(r) := \lim_{\substack{\epsilon \to 0^+ \\ v \to +\infty}} \rho_{(X,f),q}(k+\epsilon, v) - \rho_{(X,f),q}(k-\epsilon, v)$$

is strictly positive is said to be a cornerline.
$\mu(r)$ is called the multiplicity of r.



Persistence Diagrams (3)

If Čech homology is used to define persistent homology groups, then persistence diagrams completely characterize rank invariants:

Theorem (Representation Theorem)

For every $(\bar{u}, \bar{v})$ *with* $\bar{u} < \bar{v} \leq \infty$

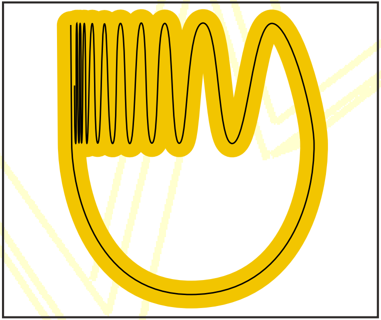
$$\check{\rho}_{(X,f),q}(\bar{u}, \bar{v}) = \sum_{\substack{(u,v):\ u<v\leq\infty \\ u\leq\bar{u},\, v>\bar{v}}} \mu((u, v)).$$

Lemma

Using Čech homology instead of singular homology, $\check{\rho}_{(X,f),q}(u, v)$ *is right-continuous w.r.t. both the variables* u *and* v*.*

Example 1

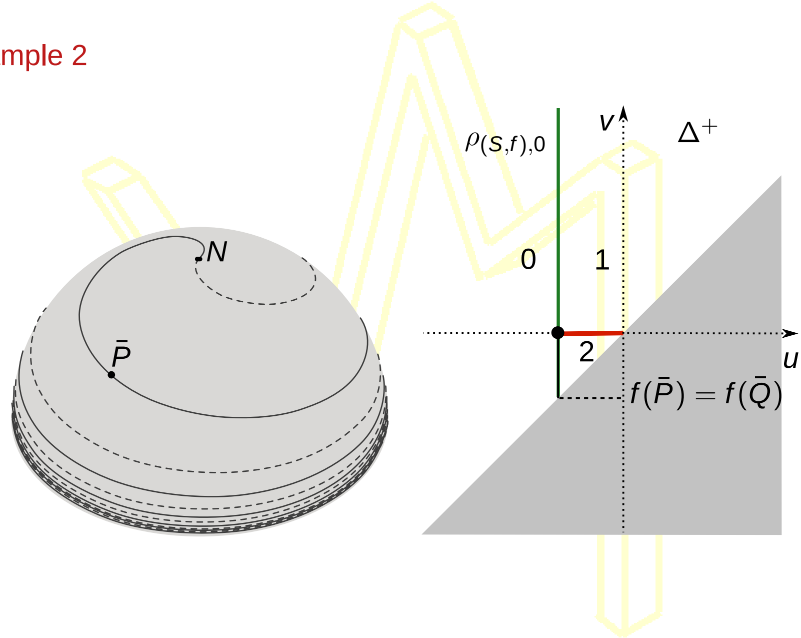
X a closed rectangle of $\mathbb{R}^2$ containing a Warsaw circle
$f : X \to \mathbb{R}$ the Euclidean distance from the Warsaw circle



$$\rho_{(X,f),1}(0,v) = 0, \quad \rho_{(X,f),1}(u,v) = 1, \ \ 0 < u < v \ \text{ suff.small},$$

$$\check{\rho}_{(X,f),1}(0,v) = 1, \quad \check{\rho}_{(X,f),1}(u,v) = 1, \ \ 0 < u < v \ \text{ suff.small}.$$

Example 2



Back to k $\geq$ 1

A suitable foliation:

$$\vec{l} = (l_1, \ldots, l_k),\ \vec{b} = (b_1, \ldots, b_k),\ \text{with } \|\vec{l}\| = 1,\ l_i > 0,\ \sum_i b_i = 0$$

- $\Delta^+ = \{(\vec{u}, \vec{v}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{u} \prec \vec{v}\}$ is foliated by the 2D half-planes with parametric equations:

$$\pi_{(\vec{l},\vec{b})} : \begin{cases} \vec{u} = s\vec{l} + \vec{b} \\ \vec{v} = t\vec{l} + \vec{b} \end{cases} \qquad s, t \in \mathbb{R}, s < t$$

- For every $(\vec{l}, \vec{b})$, define $F^{\vec{f}}_{(\vec{l},\vec{b})} : X \to \mathbb{R}$ by

$$F^{\vec{f}}_{(\vec{l},\vec{b})}(x) = \max_{i=1,\ldots,k} \left\{ \frac{f_i(x) - b_i}{l_i} \right\}.$$

Reduction to the case $\mathbf{k} = \mathbf{1}$

Reduction Theorem

For every $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ it holds that

$$\check{\rho}_{(X,\vec{f}),q}(\vec{u}, \vec{v}) = \check{\rho}_{(X,F^{\vec{f}}_{(\vec{l},\vec{b})}),q}(s, t).$$

Rank invariants are represented by persistence diagrams leaf-by-leaf

Reduction to the case $k = 1$

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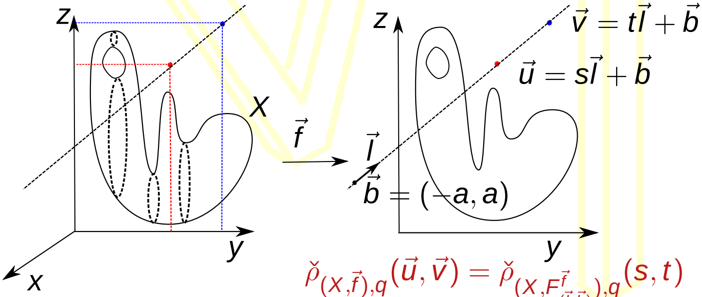

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Matching distance

Case $\mathbf{k} = \mathbf{1}$

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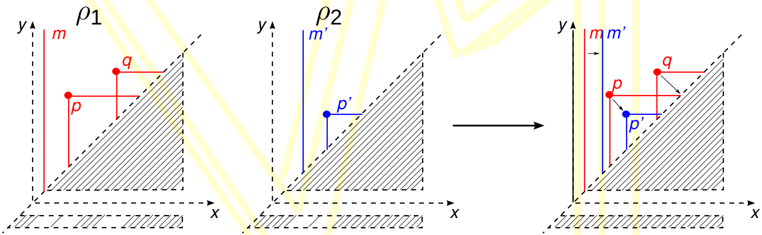
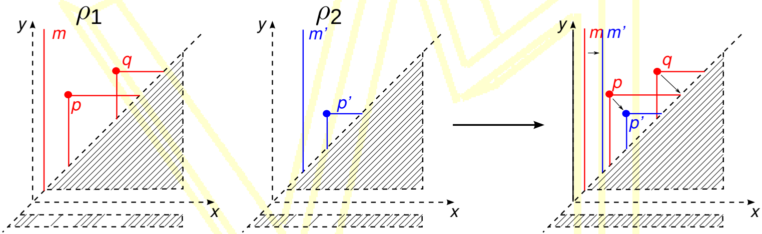


Matching distance

Case $\mathbf{k} = \mathbf{1}$



$$d_{match}(\rho_1, \rho_2) = \min_{\gamma: D \to D'} \max_{q \in D} \|q - \gamma(q)\|_\infty$$

$D = \{m, p, q, \ldots\}$, $D' = \{m', p', \ldots\}$, γ bijective.

Multidimensional Matching Distance

$$D_{match}\left(\check{\rho}_{(X,\vec{f}),q},\check{\rho}_{(X,\vec{g}),q}\right)=\sup_{(\vec{l},\vec{b})}\min_{i=1,\ldots,k} l_i\cdot d_{match}(\check{\rho}_{(X,F^{\vec{f}}_{(\vec{l},\vec{b})}),q},\check{\rho}_{(X,G^{\vec{g}}_{(\vec{l},\vec{b})}),q})$$

where

$$F^{\vec{f}}_{(\vec{l},\vec{b})}(x)=\max_{i=1,\ldots,k}\left\{\frac{f_i(x)-b_i}{l_i}\right\},\quad G^{\vec{g}}_{(\vec{l},\vec{b})}(x)=\max_{i=1,\ldots,k}\left\{\frac{g_i(x)-b_i}{l_i}\right\}.$$

X compact and locally contractible space

One-Dimensional Stability Theorem:

$f, g : X \to \mathbb{R}$ continuous functions. Then

$$d_{match}(\check{\rho}_{(X,f),q}, \check{\rho}_{(X,g),q}) \leq \max_{x \in X} |f(x) - g(x)|.$$

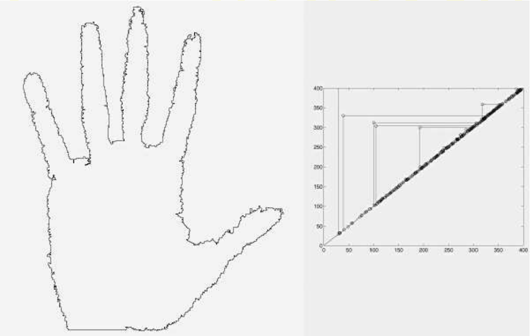
Multidimensional Stability Theorem:

$\vec{f}, \vec{g} : X \to \mathbb{R}^k$ continuous functions. Then

$$D_{match}\left(\check{\rho}_{(X,\vec{f}),q}, \check{\rho}_{(X,\vec{g}),q}\right) \leq \max_{x \in X} \|\vec{f}(x) - \vec{g}(x)\|_\infty.$$

Example of mono-dimensional stability

A curvature driven curve evolution and its 0-homology rank invariant w.r.t. the distance from the center of mass



(Thanks to Frédéric Cao for curvature evolution code)

Example of multi-dimensional stability

- Experimental results on a set of 8 human models represented by triangular meshes.
- For each model, we take the 2-dimensional measuring function $\vec{f} = (f_1, f_2)$ with

$$f_1(P_i) = 1 - \frac{||P_i - (B + \vec{w})||}{\max_j ||P_j - (B + \vec{w})||}.$$

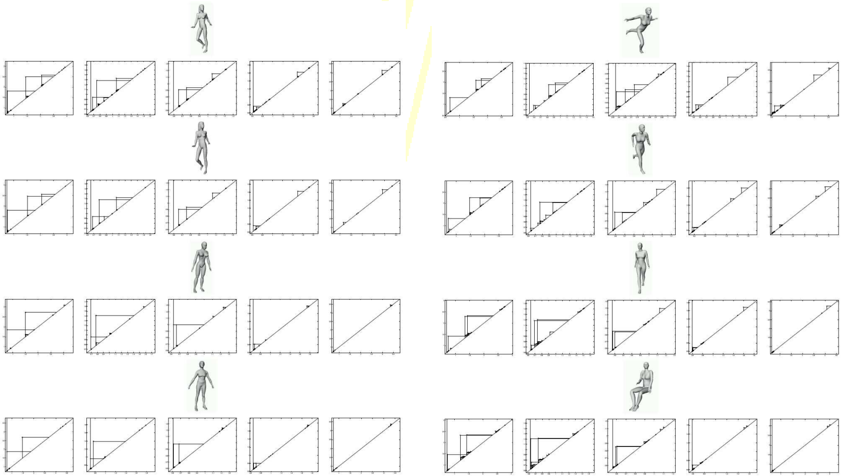
  where B is the center of mass of the model and

$$\vec{w} = \frac{\sum_{i=1}^{n} (P_i - B)||P_i - B||^2}{\sum_{i=1}^{n} ||P_i - B||^2};$$

  similarly, $f_2(P_i) = 1 - \frac{||P_i - (B - \vec{w})||}{\max_j ||P_j - (B - \vec{w})||}$.
- The admissible pairs are of the form $(\vec{l}, \vec{b})$ with $\vec{l} = (\cos\theta_i, \sin\theta_i)$, $\theta_i = \frac{\pi}{36} i$, $i = 1, \ldots, 17$, and $\vec{b} = (0, 0)$.

Results

Discretization of the 0-homology rank invariant of each model:

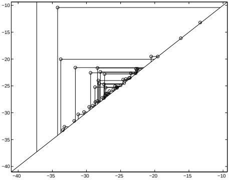


Results

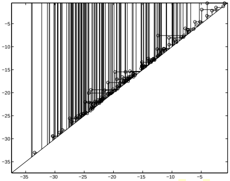
The 2D matching distance w.r.t. $\vec{f} = (f_1, f_2)$ and the max of the 1D matching distances w.r.t. f_1 and f_2:

								
	0.0000 0.0000	0.0181 0.0003	0.1411 0.0025	0.1470 0.0026	0.1325 0.0023	0.1287 0.0022	0.1171 0.0020	0.1187 0.0021
	0.0181 0.0003	0.0000 0.0000	0.1419 0.0026	0.1478 0.0026	0.1304 0.0023	0.1265 0.0022	0.1171 0.0020	0.1187 0.0021
	0.1411 0.0025	0.1419 0.0025	0.0000 0.0000	0.0137 0.0002	0.1583 0.0028	0.1370 0.0024	0.1127 0.0020	0.1017 0.0018
	0.1470 0.0026	0.1478 0.0026	0.0137 0.0002	0.0000 0.0000	0.1533 0.0027	0.1381 0.0024	0.1137 0.0020	0.1021 0.0018
	0.1325 0.0023	0.1304 0.0023	0.1583 0.0028	0.1533 0.0027	0.0000 0.0000	0.0921 0.0014	0.0588 0.0016	0.1000 0.0017
	0.1287 0.0022	0.1265 0.0022	0.1370 0.0024	0.1381 0.0024	0.0921 0.0014	0.0000 0.0000	0.1069 0.0019	0.1048 0.0018
	0.1171 0.0020	0.1171 0.0020	0.1127 0.0020	0.1137 0.0020	0.0588 0.0016	0.1069 0.0019	0.0000 0.0000	0.0350 0.0006
	0.1187 0.0021	0.1187 0.0021	0.1017 0.0018	0.1021 0.0018	0.1000 0.0017	0.1048 0.0018	0.0350 0.0006	0.0000 0.0000

K_1,K_2 = black pixels of the left and right image, resp.
$D = 72 \times 69$ pixels rectangle. Topology: 4-neighbors adjacency
$f : D \to \mathbb{R}$, $f(P) = -d(P, B)$, where B center of mass of K_1



$\rho_{(K_1, f_{|K_1}),0}$



$\rho_{(K_2, f_{|K_2}),0}$

Stability w.r.t. noisy domains $\rightsquigarrow$ stability w.r.t. noisy functions:

$$K \subseteq D \subseteq \mathbb{R}^n \rightsquigarrow d_K : D \to \mathbb{R},\ d_k(x) = \inf_{y \in K} \|y - x\|$$

$$\vec{f} : D \to \mathbb{R}^k \rightsquigarrow \vec{\Phi} : D \to \mathbb{R}^{k+1}, \vec{\Phi} = (d_k, \vec{f})$$

Theorem (Stability w.r.t. noisy domains)

Let K_1, K_2 be non-empty closed subsets of a compact and locally contractible subspace D of $\mathbb{R}^n$.
Take $\vec{\Phi}_1, \vec{\Phi}_2 : D \to \mathbb{R}^{k+1}$, $\vec{\Phi}_1 = (d_{K_1}, \vec{f}_1)$ and $\vec{\Phi}_2 = (d_{K_2}, \vec{f}_2)$. Then

$$D_{match}\left(\check{\rho}_{(D,\vec{\Phi}_1),q}, \check{\rho}_{(D,\vec{\Phi}_2),q}\right) \le \max\left\{\delta_H(K_1, K_2), \|\vec{f}_1 - \vec{f}_2\|_\infty\right\}.$$

Hausdorff distance:
$\delta_H(K_1, K_2) = \max\{\max_{x \in K_2} d_{K_1}(x), \max_{y \in K_1} d_{K_2}(y)\}$

Retrieving information about $(K, \vec{f}_{|K})$ from $(D, \vec{\Phi} = (d_k, \vec{f}))$

K a non-empty closed and locally contractible subset of a compact and locally contractible subspace D of $\mathbb{R}^n$

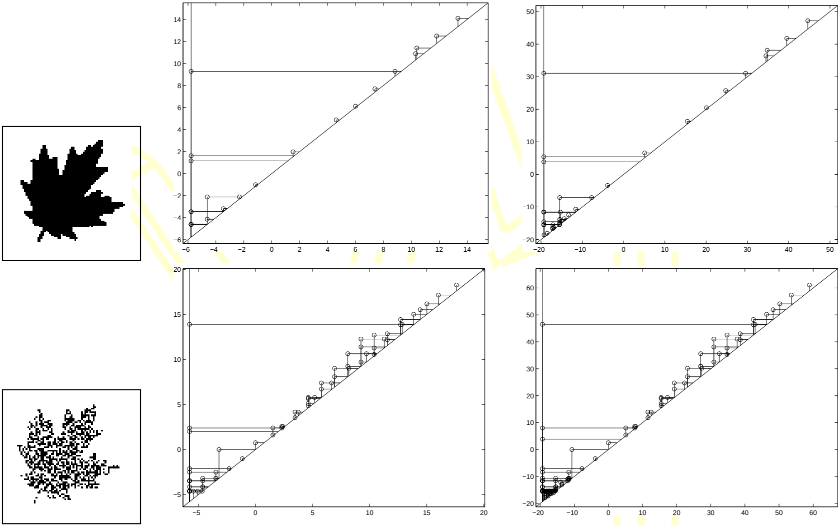
$\vec{f} : D \to \mathbb{R}^k$ a continuous function

Take $\vec{\Phi} : D \to \mathbb{R}^{k+1}$, $\vec{\Phi} = (d_K, \vec{f})$

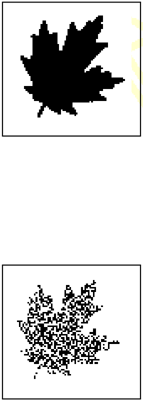
Theorem

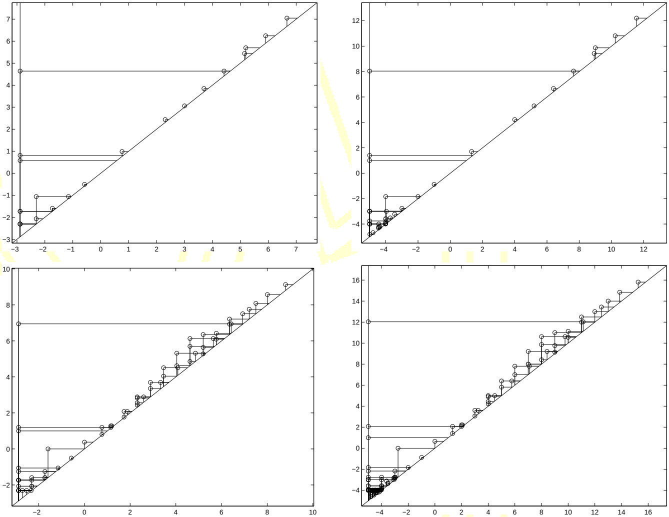
$$\check{\rho}_{(K,\vec{f}_{|K}),q}(\vec{u},\vec{v}) = \lim_{\epsilon \to 0^+} \check{\rho}_{(D,\vec{\Phi}),q}\left((0,\vec{u}),(\epsilon,\vec{v})\right),$$

for every $\vec{u},\vec{v} \in \mathbb{R}^k$ with $\vec{u} \prec \vec{v}$.



$\rho_{(D,\vec{\Phi}_1),0}$ and $\rho_{(D,\vec{\Phi}_2),0}$ restricted to the half-planes $\pi_{(\vec{l}_i,\vec{b}_i)}$ of the foliation with $\vec{l}_i = (\cos\frac{i}{12}\pi, \sin\frac{i}{12}\pi)$, $i = 2, 5$ and $\vec{b} = (5, -5)$





$\rho_{(D,\vec{\Phi}_1),0}$ and $\rho_{(D,\vec{\Phi}_2),0}$ restricted to the same half-planes $\pi_{(\vec{l}_i,\vec{b}_i)}$ of the foliation and rescaled by $\mu_i = \min_{j=1,2}(l_i)_j$.

Conclusions

- Development of a paradigm for shape comparison using
  - Multidimensional Persistence Homology
- Assessment of resistance of multidimensional homology groups under
  - noisy functions
  - noisy domains

Thank you for your attention!