(Prod-)Simplicial models for trace spaces

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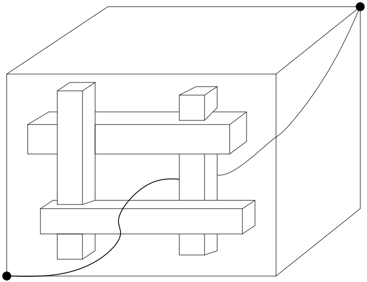
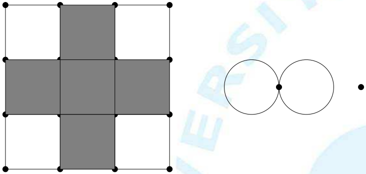
State space and model of trace space
How are they related?



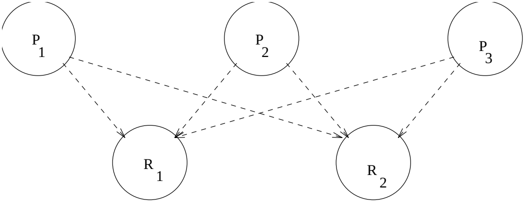
State space =
a cube minus 4 obstructions

⇝



Trace space within in a torus homotopy equivalent to a wedge of two circles and a point

Mutual exclusion occurs, when *n* processes P_i compete for *m* resources R_j.



Only *k* processes can be served at any given time.

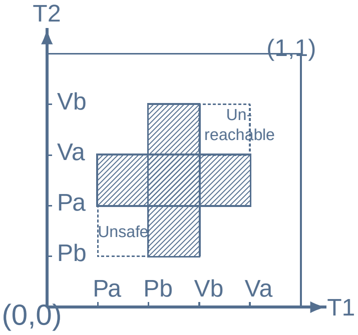
Semaphores!

Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction $P_i : \ldots PR_j \ldots VR_j \ldots$ (E.W. Dijkstra)

Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.

A linear PV-program can be modelled as the complement of a number of holes in an n-cube:
isothetic hyperrectangles $R^i, 1 \leq i \leq l$, in an n-cube:

$$X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^{l} R^i,\ R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[.$$

X inherits a partial order from $\vec{I}^n$.
More general PV-programs:

- Replace $\vec{I}^n$ by a product $\Gamma_1 \times \cdots \times \Gamma_n$ of digraphs.
- Holes have then the form $p_1^i((0,1)) \times \cdots \times p_n^i((0,1))$ with $p_j^i : \vec{I} \to \Gamma_j$ a directed (d-)path.

Spaces of d-paths/traces, Dihomotopy

X a d-space, $a, b \in X$.
$p : \vec{I} \to X$ a d-path in X (continuous and “order-preserving”)
$\vec{P}(X)(a,b) = \{p : \vec{I} \to X \mid p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
A dihomotopy on $\vec{P}(X)(a,b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a,b),\ t \in I$.
Aim: Describe the homotopy type of $\vec{P}(X)(a,b)$; in particular its path components, i.e., the dihomotopy classes of d-paths.

Covers of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

by contractible or empty subspaces

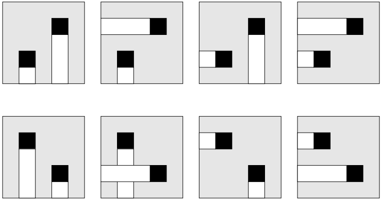
$X = \vec{I}^n \setminus F, F = \bigcup_{i=1}^{l} R^i$; $\mathbf{0},\mathbf{1}$ the two corners.

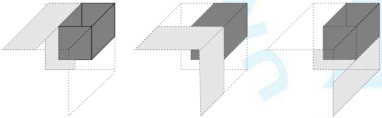
Definition

For $1 \leq j_i \leq n$ let

$$\begin{aligned} X_{j_1,\dots,j_l} &= \{x \in X \mid \forall i : x_{j_i} \leq a^i_{j_i} \lor \exists k : x_k \geq b^i_k\} \\ &= \{x \in X \mid \forall i : x \leq \mathbf{b}^i \Rightarrow x_{j_i} \leq a^i_{j_i}\} \end{aligned}$$

Examples:





$$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{1 \leq j_1,\dots,j_l \leq n} \vec{P}(X_{j_1,\dots,j_l})(\mathbf{0},\mathbf{1}).$$

Definition

For $\emptyset \neq J_1, \dots, J_l \subseteq [1:n]$ let

$$X_{J_1,\dots,J_l} = \bigcap_{j_i \in J_i} X_{j_1,\dots,j_l}$$
$$= \{x \in X \mid \forall i, j_i \in J_i : x \leq \mathbf{b}^i \Rightarrow x_{j_i} \leq a^i_{j_i}\}$$

Question: For which $J_1, \dots, J_l \subseteq [1:n]$ is $\vec{P}(X_{J_1,\dots,J_l})(\mathbf{0}, \mathbf{1}) \neq \emptyset$?

Bookkeeping with binary matrices

- $M_{l,n}$ (vector space/Boolean algebra of) binary $l \times n$-matrices
- $M^R_{l,n}$ no row vector is the zero vector
- $M^C_{l,n}$ every column vector is a unit vector

Index sets		Matrix sets
$(\mathcal{P}([1:n]))^l$	$\leftrightarrow$	$M_{l,n}$
$J = (J_1, \dots, J_l)$	$\mapsto$	$M^J = (m_{ij}),\ m_{ij} = 1 \Leftrightarrow j \in J_i$
J^M	$\leftarrow$	$M \quad J^M_i = \{j \mid m_{ij} = 1\}$
l-tuples of susbsets $\neq \emptyset$	$\leftrightarrow$	$M^R_{l,n}$
$\{(K_1, \dots, K_l) \mid [1:n] = \bigsqcup K_i\}$	$\leftrightarrow$	$M^C_{l,n}$

$$X_M := X_{J_M}, \qquad \vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \vec{P}(X_{J_M})(\mathbf{0},\mathbf{1}).$$

A combinatorial model and its geometric realization

Poset category – Combinatorics

$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^R \subseteq M_{l,n}$

$J \leftrightarrow M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

Prodsimplicial complex – Topolog[y]

$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$

$\Delta_{J_1}^{|J_1|-1} \times \cdots \times \Delta_{J_l}^{|J_l|-1} \subseteq \mathbf{T}(X)(\mathbf{0},\mathbf{1})$

$$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$$

Examples!!!

Properties of the decomposition of path space

Theorem

1. $\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{1\le i\le l, 1\le j_i\le n} \vec{P}(X_{j_1,\dots,j_l})(\mathbf{0},\mathbf{1})$.
2. *For every* $M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$*:*
   $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$ *is contractible.*

Proof.

- All $X_M, M \in M_{l,n}^R$ are closed under $\vee = \max$.
- D-homotopy $H(p,q)$ connecting $p,q \in \vec{P}(X_M)(\mathbf{0},\mathbf{1})$:
  $G(p,q): p \to p \vee q, G(q,p): q \to p \vee q,$
  $H(p,q) = G(q,p) * G^-(p,q)$
  $G(p,q;t)(s) = p(s) \vee q(ts)$
- Contraction: Choose $p \in \vec{P}(X_M)(\mathbf{0},\mathbf{1})$. $q \mapsto H(p,q)$

□

Homotopy equivalence between path space and a prodsimplicial complex

Theorem

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1}) \to \mathbf{Top}$:
  $\mathcal{D}(J_1,\dots,J_l) = \vec{P}(X_{J_1,\dots,J_l})(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(J_1,\dots,J_l) = \Delta_{J_1}^{|J_1|-1} \times \dots \times \Delta_{J_l}^{|J_l|-1}$,
  $\mathcal{T}(J_1,\dots,J_l) = *$
- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  hocolim $\mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  hocolim $\mathcal{D} \cong$ hocolim $\mathcal{T} \cong$ hocolim $\mathcal{E}$.
- Projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space
Questions answered by homology calculations

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces.
There are fast algorithms to calculate the homology groups of these chain complexes even for very big complexes.
For example: The number of path-components is the rank of the homology group in degree 0.
For path-components, there might be faster "discrete" methods.
Even if "exponential explosion" prevents precise calculations, inductive determination (round by round) of general properties ((simple) connectivity) may be possible.

Deadlocks and unsafe regions determine $\mathcal{C}(X)$

A dual view: extended hyperrectangles:
$R^i_j := [0, b^i_1[\times \cdots \times [0, b^i_{j-1}[\times]a^i_j, b^i_j[\times[0, b^i_{j+1}[\times \cdots \times [0, b^i_n[\supset R^i$.

$$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j.$$

Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
2. *There is a map* $i : [1:n] \to [1:l]$ *such that* $m_{i(j),j} = 1$ *and such that* $\bigcap_{1\le j\le n} R^{i(j)}_j \neq \emptyset$ *– giving rise to a deadlock unavoidable from* $\mathbf{0}$.
3. *Checking a bunch of inequalities:*
   There is a map $i : [1:n] \to [1:l]$ *such that*
   $a^{i(j)}_j < b^{i(k)}_j$ *for all* $1 \le j, k \le n$.

Partial orders and order ideals on matrix spaces
and an order preserving map Ψ

The partial order on $0,1$: $0 \leq 0, 0 \leq 1, 1 \leq 1$ extends to $M_{l,n}$.
Consider $\Psi : M_{l,n} \to \mathbf{Z}/2$, $\Psi(M) = 1 \Leftrightarrow \vec{P}(X_{J^M})(\mathbf{0},\mathbf{1}) = \emptyset$.

- Ψ is order preserving, in particular:
  $\Psi^{-1}(0), \Psi^{-1}(1)$ are closed in opposite senses:
  $M \leq N : \Psi(N) = 0 \Rightarrow \Psi(M) = 0, \Psi(M) = 1 \Rightarrow \Psi(N) = 1$;
  (thus $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ prodsimplicial).
- $\Psi(M) = 1 \Leftrightarrow \exists N \in M^C_{l,n}$ such that $N \leq M, \Psi(N) = 1$
  $D(X)(\mathbf{0},\mathbf{1}) = \{N \in M^C_{l,n} | \Psi(N) = 1\}$ – dead
  $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = \{M \in M^R_{l,n} | \Psi(M) = 0\}$ – alive
  $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ maximal such matrices
  characterized by: $m_{ij} = 1$ apart from:
  $\forall N \in D(X)(\mathbf{0},\mathbf{1}) \exists ! (i,j) : 0 = m_{ij} < n_{ij} = 1$ Examples!.

Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.

Which of the l^n matrices in $M^C_{l,n}$ belong to $D(X)(\mathbf{0},\mathbf{1})$?

A matrix $M \in M^C_{l,n}$ is described by a (choice) map $i:[1:n]\to[1:l], m_{i(j),j}=1$.

$M \in D(X)(\mathbf{0},\mathbf{1}) \Leftrightarrow a_j^{i(j)} < b_j^{i(k)}$ for all $1\le j,k\le n$.

Requires to check a bunch of inequalities or rather order relations.

Algorithmic organisation: Choice maps with the same image give rise to the same upper bounds b_j^*.

Algorithmics: Construct $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$ incrementally (checking for one matrix $N \in D(X)(\mathbf{0},\mathbf{1})$ at a time), starting with matrix $\mathbf{1}$:

1. $N_{i+1} \not\leq M \in \mathcal{C}^i(X) \Rightarrow M \in \mathcal{C}^{i+1}(X)$;
2. $N_{i+1} \leq M \Rightarrow M$ is replaced by n matrices M^j with one additional 0. Examples!

A matrix in $D(X)(\mathbf{0},\mathbf{1})$ describes a hyperedge on the vertex set $[1:l] \times [1:n]$; $D(X)(\mathbf{0},\mathbf{1})$ describes a hypergraph.

A transversal in a hypergraph is a vertex set that has non-empty intersection with each hyperedge

$\leftrightarrow$ a matrix L such that $\forall N \in D(X)(\mathbf{0},\mathbf{1})\ \exists(i,j) : l_{ij} = n_{ij} = 1$.

$M = \mathbf{1} - L$: $\forall N \in D(X)(\mathbf{0},\mathbf{1}) \exists(i,j) : 0 = m_{ij} < n_{ij} = 1$.

Conclusion: Search for matrices in $A_{max}(\mathbf{0},\mathbf{1})$ corresponds to search for minmal transversals in $D(X)(\mathbf{0},\mathbf{1})$.

In our case: All hyperedges have same cardinality n, include one element per column.

- More general semaphores
- $\vec{P}(X)(\mathbf{c},\mathbf{d})$ and iterative calculations

Same technique, modification of definition and calculation of $\mathcal{C}(X)$, $D(X)$ etc.

- New light on definition and determination of components.

Extensions

2a. Semaphores corresponding to non-linear programs:

$\Gamma = \prod_{j=1}^{n} \Gamma_j$, state space $X = \Gamma \setminus F, F$ product of generalized hyperrectangles R^i.

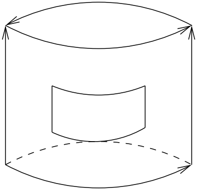
$\vec{P}(\Gamma)(\mathbf{x},\mathbf{y}) = \prod \vec{P}(\Gamma_j)(x_j,y_j)$ – homotopy discrete!

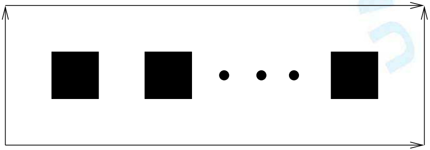
Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x},\mathbf{y})$ by (regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j,y_j)$ – an interleaving.

The map $c : \vec{I}^n \to \Gamma, c(t_1,\dots,t_n) = (c_1(t_1),\dots,c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0},\mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x},\mathbf{y})$.

Pull back F via c:

$\bar{X} = \vec{I}^n \setminus \bar{F}, \bar{F} = \bigcup \bar{R}^i, \bar{R}^i = c^{-1}(R^i)$ – honest hyperrectangles!





$i_X : \vec{P}(X) \hookrightarrow \vec{P}(\Gamma)$.
Given a component $C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.
The d-map $c : \bar{X} \to X$ induces a homeomorphism
$c\circ : \vec{P}(\bar{X}(\mathbf{0}, \mathbf{1}) \to i_X^{-1}(C) \subset \vec{P}(X)(\mathbf{x}, \mathbf{y})$.

- C "lifts to X" $\Leftrightarrow \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \neq \emptyset$; if so:
- Analyse $i_X^{-1}(C)$ via $\vec{P}(\bar{X})(\mathbf{0}, \mathbf{1})$.
- Exploit relations between various components.

- Higher Dimensional Automaton: Pre-cubical complex with preferred diretions. Geometric realization X with d-space structure.
- $P(X)(\mathbf{x},\mathbf{y})$ is ELCX (equi locally convex). D-paths within a specified "cube path" form a contractible subspace.
- $P(X)(\mathbf{x},\mathbf{y})$ has the homotopy type of a simplicial complex: the nerve of an explicit category of cube paths (with inclusions as morphisms).