Matematik og Form: Kurver i plan og rum

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Vektorfunktioner

Vektorfunktion og kurve

$\mathbf{r}(t) = (x(t), y(t), z(t))$ beskriver kurve vha. $\overrightarrow{OP_t} = \mathbf{r}(t)$.

Differentiation

$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = (x'(t), y'(t), z'(t))$ koordinatvis.
Betydning: Hastighedsvektor $\mathbf{v}(t) = \mathbf{r}'(t)$; længde
$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ svarer til farten i P_t.
Accelerationsvektoren $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$;
længde = skalar acceleration $a(t) = |\mathbf{a}(t)| \neq v'(t)$!

Integration

$\int_a^b \mathbf{r}(t)dt = (\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt)$ koordinatvis.
Anvendelse: Bestemmelse af parameterfremstilling $\mathbf{r}(t)$ med udgangspunkt i

- vandrende hastighedsvektor og begyndelsesposition, hhv.
- vandrende accelerationsvektor, samt begyndelsesposition og -hastighed.

Differentiationsregler for vektorfunktioner

Differentiationsregler

$\mathbf{u}(t), \mathbf{v}(t)$ differentiable vektorfunktioner.
$h(t)$ en (almindelig) funktion

1. $(\mathbf{u}(t) \pm \mathbf{v}(t))' = \mathbf{u}'(t) \pm \mathbf{v}'(t)$;
2. $(h(t)\mathbf{u}(t))' = h'(t)\mathbf{u}(t) + h(t)\mathbf{u}'(t)$;
3. $(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$;
4. $(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.
5. $\mathbf{u}(h(t))' = h'(t)\mathbf{u}'(h(t))$

Krumning - plane kurver; definition

- $\mathbf{T}(s) = (\cos\varphi(s), \sin\varphi(s))$ – den vandrende enhedstangentvektor.
  Retningen givet ved vinklen $\varphi(s)$ i forhold til X-aksen.
- $\mathbf{N}(s) = \hat{\mathbf{T}}(s) = (-\sin\varphi(s), \cos\varphi(s))$ – den vandrende enhedsnormalvektor.
- $\mathbf{T}'(s)$ måler vinkelhastigheden $\varphi'(s)$ og er vinkelret på $\mathbf{T}(s)$:
  - $\mathbf{T}'(s) = \varphi'(s)(-\sin\varphi(s), \cos\varphi(s)) = \varphi'(s)\mathbf{N}(s)$

Definition

Krumning $\kappa(s) = \varphi'(s)$ i $\mathbf{r}(s)$:

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s).$$

Græske bogstaver

φ	phi
κ	kappa

Krumning - plane kurver; beregning

- $\mathbf{v}(t) = \mathbf{r}'(t) = v(t)\mathbf{T}(t)$
- $\mathbf{T}'(t) = \frac{d}{dt}\mathbf{T}(t) = \frac{d}{ds}\mathbf{T}(t) \cdot \frac{ds}{dt} = \kappa(t)v(t)\mathbf{N}(t)$
- $\mathbf{a}(t) = \mathbf{r}''(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) =$ $v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t)$ – tangential og normal komponent af accelerationsvektoren
- Projektionen på normalvektoren ved prikprodukt med $\mathbf{N}(t)$: $\kappa(t)v^2(t) = \mathbf{a}(t) \cdot \mathbf{N}(t) = \mathbf{a}(t) \cdot \frac{1}{v(t)}\hat{\mathbf{r}}'(t)$
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$$\begin{aligned}
\kappa(t) &= \frac{1}{v^3(t)}\mathbf{r}''(t) \cdot \hat{\mathbf{r}}'(t) \\
&= \frac{1}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}(x''(t), y''(t)) \cdot (-y'(t), x'(t)) \\
&= \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}
\end{aligned}$$

Rumkurver

I rummet er der en hel plan af normalretninger i hvert punkt.
Krumning $\kappa(s) = |\mathbf{T}'(s)|$.
Hovednormalvektor $\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\kappa(s)}$.

En formel for krumningen κ

$$\begin{aligned}
\mathbf{a}(t) = \mathbf{r}''(t) &= v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \\
&= v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t) \\
\mathbf{v}(T) \times \mathbf{a}(t) &= v(t)\mathbf{T}(t) \times (v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t)) \\
&= \mathbf{0} + v^3(t)\kappa(t)(\mathbf{T}(t) \times \mathbf{N}(t)) \\
|\mathbf{v}(T) \times \mathbf{a}(t)| &= v^3(t)\kappa(t) \\
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.
\end{aligned} \tag{1}$$