This is an activity of type 5: no lecturer available! Instead, please take departure in the list of suggestions (both reading texts and working with exercises) below. Questions you might wish to ask the lecturer: during the exercise session of the next activity.

Inverse and implicit function theorem

These two theorems will be important tools several times during this course. We will not insist on the proofs, but you will need to have the statements and their meaning present.

Please repeat the two theorems from the course Analysis 2 (see references below). The inverse function theorem gives you conditions under which a differentiable function on an open set has a *local* inverse – generalising what you know about invertibility of linear maps, right?

The implicit function theorem will be particularly important for the case of a *real* function h on an open set $U \subseteq \mathbf{R}^m$, $h : U \to \mathbf{R}$ – $\mathbf{R} = \mathbf{R}^1$! In other words, we look at solutions of an equation of type $h(x) = c \in \mathbf{R}$ and try to express them as the *graph* of a function of $m-1$ variables. Formulate conditions as well as conclusions in this particular case explicitly.

Exercises

1. The complex squaring map $F : \mathbf{C} \to \mathbf{C}$ can be written in real coordinates as $F(x,y) = (x^2 - y^2, 2xy)$. Check the conditions of the inverse function theorem (where are they satisfied, where not?) and draw conclusions about complex square root maps.

2. Investigate the maps

   (a) $h : \mathbf{R}^2 \to \mathbf{R}, h(x,y) = xy$

   (b) $k : \mathbf{R}^3 \to \mathbf{R}, k(x,y,z) = xyz$ in the light of the implicit function theorem. Where are the conditions satisfied, where not? What can you conclude about solutions of equations $xy = c,\ c \in \mathbf{R}$, resp. $xyz = c,\ c \in \mathbf{R}$? How does $c = 0$ differ from the other cases?

References

Wade W.R. Wade, *An introduction to anlaysis*, Ch. 11.6; in particular Thm. 11.41, 11.47

Cornean H. Cornean, Notes for the course Analyse 2, Ch. 3, 4; in particular Thm. 3.7, 4.3

Pressley A. Pressley, *Elementary Differential Geometry*, Ch. 5.6, in part. Thm. 5.6.1

Wikipedia Inverse function theorem
Implicit function theorem

Existence and uniqueness of solutions of first order ODE's

Many results in dfferential geometry rely on the fact that (systems of) 1st order differential equations $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t))$ have unique solutions (given an initial value and that f is decent, eg Lispschitz). We will meet applications of this fact for the first time in the proof of the main theorem of curve theory (Thm. 2.3.6 in Pressley). This is why I suggest you to repeat the theorem on existence and uniqueness of solutions of ODEs (at least the conditions and the result). Observe that, in general, solutions exist on some maximal open interval, not necessarily on the whole real line.

In fact, we will need a sharpened version of that theorem, for systems of *linear* ODEs of the form $\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{f}(t)$. If the matrix function $A(t)$ and the vector function $\mathbf{f}(t)$ are continuous on the real line, then every initial value problem has a unique solution defined on the real line ("lives for ever"). Briefly, for every initial value $\mathbf{y}_0 \in \mathbf{R}^n$, the right hand side $A(t)\mathbf{y}_0 + f(t)$ is *bounded* on every closed real interval. This is why one can *extend* a solution on an open interval to its boundary, and then extend it from there to a larger interval. Loosely speaking, a solution to a system of linear ODEs will never "go to infinity". Try to make sense of these lines!

References

Cornean H. Cornean, Notes for the course Analyse 2, Ch. 2.2. The system of differential equations and the initial value problem are hidden in equation (2.8).

Wikipedia Initial value problem

Exercises

You might still have some exercises left from Activity 1. Try those first! Further suggesions:

- Pressley, Ex. 1.5.2-3 (p. 27)
- Pressley, Exercise 5.6.1 (p. 119).