Differential Geometry
Fundamental forms. Curvature notions on surfaces.

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Definition

The first fundamental form on the tangent space $T_{\mathbf{p}}S$ on a surface S at $\mathbf{p} \in S$ given by

$$< \mathbf{v}, \mathbf{w} >_I = \mathbf{v} \cdot \mathbf{w}, \ \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$$

is a symmetric non-degenerate bilinear form.

Gram matrix in local coordinates

Let $\sigma : U \to S \subset \mathbf{R}^3$ and let $E, F, G : U \to \mathbf{R}$ denote the functions(!)

$$E = \sigma_u \cdot \sigma_u, \ F = \sigma_u \cdot \sigma_v, \ G = \sigma_v \cdot \sigma_v.$$

For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\sigma_u + v_2\sigma_v, \ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1\sigma_u + w_2\sigma_v,$:

$$< \mathbf{v}, \mathbf{w} >_I = \mathbf{v}^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} \mathbf{w}.$$

Let $\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ denote the Gram matrix corresponding to $<,>_I$ with respect to a local coordinate system σ. It allows to calculate

Curve length $L_{t_0}^{t_1}(\sigma(u(t), v(t))) = \int_{t_0}^{t_1} ||\frac{d}{dt}\sigma(u(t), v(t))|| =$

$$\int_{t_0}^{t_1} \sqrt{\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}^T \mathcal{F}_I \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}} dt = \int_{t_0}^{t_1} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} dt$$

Angles $\cos\angle(\mathbf{v} \cdot \mathbf{w}) = \dfrac{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T \mathcal{F}_I \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}{\sqrt{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T \mathcal{F}_I \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}^T \mathcal{F}_I \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}}$

Surface area $\text{area}(\sigma(R)) = \iint\limits_R \sqrt{EG - F^2} dudv$

(Local) isometries

Definition

1. A linear map $T : V \to W$ between vector spaces with bilinear forms is called an isometry if
$< \mathbf{v}, \mathbf{w} >_V = < T(\mathbf{v}), T(\mathbf{w}) >_W$ for all $\mathbf{v}, \mathbf{w} \in V$.
2. A smooth map $f : S_1 \to S_2$ is a local isometry if
$< \mathbf{v}, \mathbf{w} >_I = < D_p f(\mathbf{v}), D_p(\mathbf{w}) >_I$ for all $p \in S_1, \mathbf{v}, \mathbf{w} \in T_p S_1$
(and hence $D_p f(\mathbf{v}), D_p f(\mathbf{w}) \in T_{f(p)} S_2$).

Theorem

1. *A linear isometry preserves lengths of and angles between vectors and areas/volumes.*
2. *A local isometry preserves lengths of and angles between curves and areas of surface regions.*
3. *A local isometry is (automatically) also a local diffeomorphism.*

Definition

A local isometry $f : S_1 \to S_2$ is an isometry if it is a diffeomorphism.

Definition

Given an oriented surface S

- The Gauss map $\mathcal{G} : S \to S^2$ associates to every point $\mathbf{p} \in S$ the unit normal vector $\mathbf{N}(\mathbf{p}) \in S^2$.
- The Weingarten map:
  $W_{\mathbf{p}} := -D_{\mathbf{p}}\mathcal{G} : T_{\mathbf{p}}S \to T_{\mathcal{G}(\mathbf{p})}S^2 = \mathbf{N}(\mathbf{p})^{\perp} = T_{\mathbf{p}}S$
  at $\mathbf{p} \in S$ is a linear self-map.
- The second fundamental form on $T_{\mathbf{p}}S$, given by

$$< \mathbf{v}, \mathbf{w} >_{II} = < W_{\mathbf{p}}(\mathbf{v}), \mathbf{w} >_{I}$$

  is a symmetric bilinear form, i.e., $W_{\mathbf{p}}$ is self-adjoint wrt. $<, >_I$.

With respect to a regular coordinate patch $\sigma : U \to S$ with $\sigma(\mathbf{q}) = \mathbf{p}$ and the associated basis $\{\sigma_u(\mathbf{q}), \sigma_v(\mathbf{q})\}$ for $T_{\mathbf{p}}S$, the second fundamental form $<,>_{II}$ on $T_{\mathbf{p}}S$ has the following description in terms of its Gram matrix $\mathcal{F}_{II}$:

$$< \mathbf{v}, \mathbf{w} >_{II} = \mathbf{v}^T \mathcal{F}_{II} \mathbf{w} = \mathbf{v}^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} \mathbf{w}$$

with $L = \mathbf{N} \cdot \sigma_{uu}, M = \mathbf{N} \cdot \sigma_{uv} = \mathbf{N} \cdot \sigma_{vu}, N = \mathbf{N} \cdot \sigma_{vv}$.
The coefficients E, F, G, L, M, N of the two fundamental forms may be considered as smooth functions on the coordinate domain $U \subseteq \mathbf{R}^2$.

Normal curvature and geodesic curvature

The curvature of a smooth curve γ on a surface S at $\mathbf{p} \in S$ decomposes into the normal curvature κ_n and the geodesic curvature κ_g:

$$\begin{aligned}
\kappa\mathbf{n} = \ddot{\gamma} &= (\ddot{\gamma} \cdot \mathbf{N})\mathbf{N} + (\ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma}))(\mathbf{N} \times \dot{\gamma}) \\
&= \kappa_n \mathbf{N} + \kappa_g (\mathbf{N} \times \dot{\gamma}) \\
\kappa^2 &= \kappa_n{}^2 + \kappa_g{}^2 \\
\kappa_n &= \kappa \cos \angle(\ddot{\gamma}, \mathbf{N})
\end{aligned}$$

Theorem (Meusnier's theorem)

The normal curvature κ_n depends only on the tangent direction $\mathbf{t} = \dot{\gamma}$ since:

$$\kappa_n(\mathbf{t}) = \ddot{\gamma} \cdot \mathbf{N} \stackrel{!}{=} -\dot{\gamma} \cdot \dot{\mathbf{N}} = < -\dot{\mathbf{N}}, \dot{\gamma} >_I =$$
$$= < W_{\mathbf{p}}(\dot{\gamma}), \dot{\gamma} >_I = < \dot{\gamma}, \dot{\gamma} >_{II} = < \mathbf{t}, \mathbf{t} >_{II}.$$

The principal curvatures κ_1 and κ_2 are the maximal, resp. minimal normal curvatures at a point $p \in S$ (compared to all other tangent directions at p). They are the eigenvalues of the Weingarten map W_p corresponding to an orthonormal basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ of eigenvectors (W_p is self-adjoint, spectral theorem!)

Theorem (Euler's formula)

$$\kappa_n(\mathbf{p}; \mathbf{t}) = \kappa_1(\mathbf{p}) \cos^2(\theta) + \kappa_2(\mathbf{p}) \sin^2(\theta), \; \theta = \angle(\mathbf{t}_1, \mathbf{t}).$$

Weingarten map: wrt. basis $\{\mathbf{t}_1, \mathbf{t}_2\}$: $W_\mathbf{p} = \begin{bmatrix} \kappa_1(\mathbf{p}) & 0 \\ 0 & \kappa_2(\mathbf{p}) \end{bmatrix}$

Gaussian curvature: $\mathbf{K}(\mathbf{p}) = \det W_\mathbf{p} = \kappa_1(\mathbf{p})\kappa_2(\mathbf{p})$

Mean curvature: $\mathbf{H}(\mathbf{p}) = \frac{1}{2} \operatorname{Tr} W_\mathbf{p} = \frac{\kappa_1(\mathbf{p}) + \kappa_2(\mathbf{p})}{2}$

Calculation of curvatures

With respect to a chart basis $\{\sigma_u(\mathbf{q}), \sigma_v(\mathbf{q})\}$, $\sigma(\mathbf{q}) = \mathbf{p}$, we get:

Weingarten map: $W_{\mathbf{p}} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$ (at $\mathbf{q}$)

Gaussian curvature: $\mathbf{K}(\mathbf{p}) = \det W_{\mathbf{p}} = \frac{LN-M^2}{EG-F^2}$ (at $\mathbf{q}$)

Mean curvature: $\mathbf{H}(\mathbf{p}) = \frac{1}{2} \operatorname{Tr} W_{\mathbf{p}} = \frac{GL+EN-2FM}{2(EG-F^2)}$ (at $\mathbf{q}$)

Principal curvatures: $\kappa_1(\mathbf{p}), \kappa_2(\mathbf{p})$ = roots in $\kappa^2 - 2H(\mathbf{p})\kappa + K(\mathbf{p}) = 0$,

$$\kappa_i(\mathbf{p}) = H(\mathbf{p}) \pm \sqrt{H^2(\mathbf{p}) - K(\mathbf{p})}$$

Principal directions: given by vectors $\mathbf{t}_i$ in the kernel of the rank 1 matrix

$$\begin{bmatrix} L - \kappa_i E & M - \kappa_i F \\ M - \kappa_i F & N - \kappa_i G \end{bmatrix} \text{ (at } \mathbf{q}).$$