Quotients

When? Wed., September 15; 8:45 – 11:45
Where? Fredrik Bajersvej 7G 5-109

Lectures

Aims and Content

A manifold can be considered as a collection of pieces of $\mathbb{R}^n$ which are glued together - a choice of a set of charts covering a manifold will give this picture. A quotient is a different construction that may yield a manifold, as well:

Given an equivalence relation on a manifold, the set of *equivalence classes* with the *quotient topology* is a topological space. But even if the original space is Hausdorff and second countable, the quotient sometimes is not; and then it surely does not support a manifold structure. However, in prominent examples, a manifold does emerge from a quotient. In particular, we will study projective space, $\mathbb{R}P^n$, which is the set of *lines* in $\mathbb{R}^{n+1}$; very useful in eg robotics. Projective space can be described as a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ under the relation $p \sim tp$ for $t \in \mathbb{R} \setminus \{0\}$. In general, when considering quantities, which are invariant under scaling, $F(p) = F(tp)$, the proper space to work in is projective space.

A similar construction is the set $G(k,n)$ of k-dimensional subspaces of $\mathbb{R}^n$, the Grassmannian. Grassmannians and projective spaces, which are Grassmannians $G(1,n)$, play a role in physics, for instance in Yang Mills theory.

It is possible to describe these manifolds using charts, but this is much more complicated then using a description as a quotient.

Lecturer:

Lisbeth Fajstrup

References:

[LWT] Ch. 7. Moreover, the definition of an equivalence relation, chapter 2, the first few lines of section 2.2.
OBS: In many places, a quotient $S/\sim$ has become $S \not\sim$ in the book. The author seems to have problems with his hspace commands.

FREDRIK BAJERSVEJ 7
9220 AALBORG ØST
9940 8848 FAJSTRUP@MATH.AAU.DK
9940 8855 RAUSSEN@MATH.AAU.DK
9940 8762 RAF@ES.AAU.DK

LISBETH FAJSTRUP
MARTIN RAUSSEN
RAFAEL WISNIEWSKI

Exercises:

- (LWT) Exercise 7.11 p.68
- With reference to p. 71-72, calculate $\phi_3 \circ \phi_4^{-1}$.
- Let us study Grassmannians without worrying too much about the topology, i.e., beginning on p. 73 read the last three lines and go through the construction of an atlas on $G(2,4)$ on p. 74. You have to read the definition of $F(k,n)$ and the equivalence relation on p. 73 as well. The rank of a matrix has many equivalent definitions. The one you know may remember from a 1st year linear algebra course is "the number of Pivot entrances in a row equivalent echelon matrix".

Tangent vectors in $\mathbb{R}^n$ as derivations

Wed, September 15; 12:30 – 15:30

Lectures

Aims and Content

To do calculus on manifolds – linear approximations etc. – we need to define tangent spaces, and we prepare this approach with a second look at the situation in $\mathbb{R}^n$: We are used to think of a tangent vector as a short arrow sitting inside $\mathbb{R}^n$. In fact, it is hard to see the difference between $\mathbb{R}^n$ and the space $T_p\mathbb{R}^n$ of tangents at a point $p \in \mathbb{R}^n$. Indeed, the tangent space is an n-dimensional vector space, and hence isomorphic to $\mathbb{R}^n$.

A manifold does not come with ambient space, and thus it is not clear where "velocity vectors" would live. We prepare another view by giving a different description for tangent vectors in $\mathbb{R}^n$. Instead of what they are, we will focus on what they "do". I.e., the information contained in a tangent vector is reflected in the operation *directional derivative* along the tangent vector performed on smooth functions.

The general framework is that of *derivations* on function spaces or rather on spaces of *germs*: Two smooth functions which agree in a small neighborhood of $p \in \mathbb{R}^n$, have the same directional derivative at p. A function f defined in a neighborhood U of p is denoted (U, f). Two such locally defined functions, (U, f) and (V, g) are equivalent at p , if there is a neighborhood $p \in W \subset (U \cap V)$ such that $f_{|W} = g_{|W}$: the restrictions agree. The equivalence classes are the germs at p, $C_p^\infty(\mathbb{R}^n)$. A tangent vector is a *derivation* on germs, i.e., a linear map from germs to $\mathbb{R}$, which satisfies the *Leibniz rule* - the usual product rule for differentiation.

FREDRIK BAJERSVEJ 7
9220 AALBORG ØST
9940 8848
9940 8855
9940 8762
FAJSTRUP@MATH.AAU.DK
RAUSSEN@MATH.AAU.DK
RAF@ES.AAU.DK

By the way: There are many definitions of tangent spaces to cater for different needs. Here we consider C^∞ functions and not C^k for $1 \leq k < \infty$ and this gives some advantages. The afficionados may take a look at the proof of Theorem 2.3 (the only proof with some meet in this section). The functions g_i in that proof are C^∞. If f were C^k, they would be C^{k-1}, and the proof would fall apart.

For other definitions, see the literature listed on the course webpage. All definitions work for C^∞, but some may not work for analytic manifolds and others may not work for C^k, $k < \infty$.

Lecturer:

Martin Raussen

References:

[LWT], Ch. 2 sections 1 – 3.

Exercises:

- (LWT) p.18, Exercise 2.2 – 2.4 (the last one is more interesting than the others!)
- The exercises left from this morning.

FREDRIK BAJERSVEJ 7
9220 AALBORG ØST
9940 8848
9940 8855
9940 8762
FAJSTRUP@MATH.AAU.DK
RAUSSEN@MATH.AAU.DK
RAF@ES.AAU.DK