The Tangent Space

When?
Fri., September 17; 8:45 – 11:45
Where? Fredrik Bajers Vej 7G5-109

Lectures

Aims and Content

At every point of a smooth manifold M^m there is a tangent space, an m-dimensional *vector space*. The tangent space is the space of derivations on $C_p^\infty(M)$, the algebra of germs of smooth functions. Given a smooth map $F : M \to N$ between two manifolds, the *differential* of F at $p \in M$ is a linear transformation $DF_p : T_pM \to T_{F(p)}N$, which generalizes the derivative or Jacobi matrix of a smooth map between Euclidean spaces.

There is also a chain rule for such differentials with the following consequence: If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets of two Euclidean spaces and if there is a diffeomorphism $\psi : U \to V$, then $n = m$, i.e., the dimensions are the same. Hence, the dimension of a smooth connected manifold is well defined - all charts map to Euclidean spaces of the same dimension.

Given a chart (U, ϕ) on M^n, we obtain a basis for the vector space T_pM, consisting of the partial derivatives $\frac{\partial}{\partial x_j}|_p$. This is slang for taking the usual partial derivatives of $f \circ \phi^{-1}$ for a smooth function $f : M \to \mathbb{R}$. Given two charts (U, ϕ) and (V, ψ) on M around p, the change of coordinates matrix is the Jacobi matrix for the map $\psi \circ \phi^{-1}$, and more generally, given a smooth map $F : M \to N$, charts (U, ϕ) around $p \in M$ and (V, ψ) around $F(p)$, the matrix for DF_p with respect to the bases given by the two charts, is the Jacobi matrix for the map $\psi \circ F \circ \phi^{-1}$.

For a smooth curve $\gamma :]a, b[\to M$ on a manifold, the velocity vectors are tangent vectors: For $t \in]a, b[$, $\gamma'(t)$ is the derivation $\gamma'(t)(f) = (f \circ \gamma)'(t)$. In fact, *all* derivations can be expressed this way: Given $X_p \in T_pM$, there is a smooth curve $\mu :] - \epsilon, \epsilon[\to M$ such that $\mu(0) = p$ and $X_p(f) = (f \circ \mu)'(0)$ for all germs (U, f). This gives another description of T_pM - as the derivatives along curves.

Lecturer:

Lisbeth Fajstrup

References:

[LWT] Ch. 8.

Exercises:

[LWT], ch. 8, pp. 87 – 89.

- (LWT) 8.1, 8.3,
- We define *stereographic projection* $\sigma : S^n \setminus \{(0, 0, \cdots, -1)\} \to \mathbb{R}^n$, where S^n is the unit n-sphere $\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\}$, by

$$\sigma(x_1, \cdots, x_{n+1}) = \frac{(x_1, \cdots, x_n)}{1 - x_{n+1}}$$

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Prove that σ is a smooth map. Calculate the matrix of $D\sigma_{(x_1,\dots,x_{n+1})}$ at a point in the hemisphere $U = \{p \in S^n | x_{n+1} > 0\}$ wrt. the chart $(x_1, \dots, x_{n+1}) \to (x_1, \dots, x_n)$ on U. (The chart on the image $\mathbb{R}^n$ of σ is the identity map, but you probably guessed that.)

- (LWT) 8.6

Submanifolds

Fri, September 17; 12:30 – 15:30

Lectures

Aims and Content

During this lecture, we will introduce the notion of a regular *submanifold* (of a "big" manifold). Informally, this is a subset of this "big" manifold with the property that locally some of the coordinate functions vanish. A prototype for a submanifold is the xy-plane in $\mathbb{R}^3$, where the z-coordinate function vanishes.

We will introduce the *Regular Level Set Theorem*, which says that if $f : N \to M$ is a smooth map between two manifolds M and N and $p \in M$ is a *regular value* then the inverse image of p is a submanifold of N of dimension $\dim(N) - \dim(M)$.

The inverse image of a point p can be interpreted (locally) as the solution to a system of equations expressed by the map f, i.e., $\{x \in N | f(x) = p\}$. In conclusion, the Regular Level Set Theorem names conditions assuring that the solution to a system of equations is a manifold. In many cases, this is *the* way to prove that a certain space has a manifold structure!

Lecturer:

Rafael Wisniewski

References:

[LWT] Ch. 9.

Exercises:

[LWT], ch. 9, pp. 98 – 100.

- 9.1 – 9.4