Lie Groups

When? Thu, September 27
8:45 – 11:45
Where? FrB7G5-109

Lectures

Aims and Content

As we discussed in the course, some manifolds have the structure of a group. A Lie Group is a smooth manifold which is also a group and such that the two group operations, multiplication and inversion, are smooth maps.

The focus in the lecture will be on matrix Lie groups. We already studied some examples of Lie Groups: the general linear group

$$GL(n, \mathbf{R}) = \{A \in \mathbf{R}^{n\times n} |\ \det A \neq 0\},$$

the special linear group

$$SL(n, \mathbf{R}) = \{A \in \mathbf{R}^{n\times n} |\ \det A = 1\}$$

and the orthogonal group

$$O(n) = \{A \in \mathbf{R}^{n\times n} |\ AA^T = I\}.$$

The latter are subgroups and closed subsets of $GL(n, \mathbf{R})$. Such a group is called a closed subgroup.

An important result of the lecture is that a closed subgroup of a Lie group is an embedded submanifold.

The second topic of the lecture is concerned with the exponential map on square matrices: its definition and its role as solution of the matrix differential equation $Y' = XY$. Exponential map, determinant and trace of a matrix are connected through the equation $\det(e^X) = e^{\mathrm{tr}X}$. This is the key to the determination of the differential $\det_{*,I}$ of the determinant map at I: It maps a matrix to its trace. As a consequence,

$$T_I(Sl(n, \mathbf{R})) = \{A \in \mathbf{R}^{n\times n} |\ \mathrm{tr} A = 0\}.$$

Lecturer:

Martin Raussen

References:

[LWT] ch. 15.

Exercises:

- **LWT, ch. 14, pp. 144 - 146:** "Leftovers" from the last session
- **LWT, ch. 15, pp. 157 - 160:** 15.4, 15.5, 15.8, 15.10.

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Lie Algebras

Mon, September 29; 12:30 – 15:30

Lectures

Aims and Content

The Lie algebra $\mathfrak{g}$ of a Lie group G is its tangent space at the identity e. For $SL(n, \mathbf{R})$, it can be identified with the vector space of $n \times n$-matrices of trace 0; for $O(n)$, it corresponds to the vector space of skew-symmetric $n \times n$-matrices.

In general, $\mathfrak{g}$ can be identified with the space of left invariant vector fields $L(G)$ on G. From these, it inherits more algebraic structure: the Lie bracket $[\ ,\ ]$. If G is a matrix group – as in the cases above – this bracket is the bracket defined on square matrices by the (anti-commutative) formula $[A, B] = AB - BA$.

It turns out, that the differential F_* of a Lie group homomorphism $F : G \to H$ between two Lie groups respects the Lie brackets on the two Lie algebras.

Lecturer:

Rafael Wisniewski

References:

[LWT], ch. 16.

Exercises:

- **LWT, ch. 15, pp. 157 – 160:** 15.6 – 7, 11 – 13.

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