Homework set 1
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The Lie group $SO(3)$ consists of all orthogonal 3×3-matrices with determinant 1 ($AA^T = I_3, \det A = 1$). It contains the length and orientation preserving linear transformations in $\mathbb{R}^3$ and is essential in both mechanics and robotics. $SO(3)$ is a manifold since $SO(3) \subset O(3) = G(3,3)$. For applications, it is desirable to manage it using a manifold with a simpler description. This is where the *quaternions* H come in:

$H := \{a + bi + cj + dk |\ a, b, c, d \in \mathbb{R}\}$ is a 4-dimensional vector space with a (non-commutative!) multiplication

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j \qquad (1)$$

that extends linearly to all of H.
There is a conjugation map $z = a + bi + cj + dk \mapsto \bar{z} = a - bi - cj - dk$, and it is easy to verify that

1. $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$
2. $z\bar{z} = \bar{z}z = \| z \|^2 1$

with $\|\|$ denoting Euclidean length. Elements $z \neq 0$ have therefore a multiplicative *inverse* $z^{-1} = \frac{\bar{z}}{\|z\|}$. Remark that, with the restricted multiplication, $S^3 = \{z \in H |\ \| z \| = 1\} = \{z \in H |\ z\bar{z} = 1\}$ becomes a (non-commutative) Lie group.

The imaginary elements $z \in H$ fill the 3-dimensional subspace $Im(H) = \{bi + cj + dk |\ b, c, d \in \mathbb{R}\} \subset H$ with basis i, j, k; an element $z \in H$ is imaginary if and only if it satisfies the equation $\bar{z} = -z$.

More on quaternions

Have a look at Wikipedia.

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1. Show: The map $\varphi : H \to M(4, \mathbb{R}),\ \varphi(a + bi + cj + dk) = \begin{bmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{bmatrix}$ is a multiplicative homomorphism (i.e., $\varphi(z_1 z_2) = \varphi(z_1)\varphi(z_2)$) – it is enough to check this on the basis $1, i, j, k$ of H – with the additional property $\varphi(\bar{z}) = \varphi(z)^T$.
As a consequence: φ describes a Lie group homomorphism $\varphi : S^3 \to SO(4)$ (defined like $SO(3)$ above, but for 4×4-matrices).

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2. Show: Every element $x \in H$ defines a linear map

$$\psi_x : Im(H) \to Im(H),\ \psi_x(u) = xu\bar{x}.$$

Why linear, why is the result contained in Im (H)?
Moreover: For $x \in S^3$ and $u \in Im(H)$, one has: $\| \psi_x(u) \| = \| u \|$.
Hence ψ_x can be viewed as an element of $O(3)$.

3. Conclude that the maps ψ_x altogether define a map $\psi : S^3 \to SO(3)$.
To check, that $\det \psi_x = 1$ for all x, one may start with $\psi_1 = I_3$; $\det \circ \psi$ is continuous; orthogonal matrices have determinant ± 1.
Check that ψ is a *homomorphism* (i.e., $\psi(z_1z_2) = \psi(z_1)\psi(z_2)$) with kernel $\{x \in S^3 | \ \psi_x = I_3\} = \{\pm 1\}$. Hence, ψ factors to yield a smooth and one-to-one map $\bar{\psi} : \mathbb{R}P(3) \to SO(3)$.

4. Check the following matrix representation for $\psi_{a+bi+cj+dk} \in SO(3)$ for $a + bi + cj + ck \in S^3$:

$$\psi_{a+bi+cj+dk} = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}$$

(Hint: The columns are the components of $\psi_{a+bi+cj+dk}(u)$ with $u = i, j, k$.)

For your information (not part of the homework):

1. The map ψ covers all of $SO(3)$ (is surjective) and can thus it parametrizes $SO(3)$. As a consequence, $\bar{\psi} : \mathbb{R}P(3) \to SO(3)$ is a diffeomorphism identifying these two manifolds!

2. Similarly, one may define a map $\eta : S^3 \times S^3 \to SO(4)$ where $SO(4)$ describes the orthogonal maps on *all* of H (instead of $Im(H)$). It is given by $\eta_{x,y}(u) = xu\bar{y}$ and η has kernel $\{\pm(1,1)\}$. Hence, $SO(4)$ is a 6-dimensional smooth manifold, diffeomorphic to the quotient of $S^3 \times S^3$ by the equivalence relation $(x, y) \sim (-x, -y)$.

3. For alternative descriptions of $SO(3)$ have a look at various Wikipedia pages:

   - Charts on $SO(3)$
   - Euler angles

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- Maps of constant rank – the second example.

We ask you to work out solutions to the questions/exercises above, preferably in groups of two or three participants.
Please hand your solutions in, either on paper or electronically, no later than Monday, September 27 (the last day of the course).

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