Differential Geometry

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A vector space consists of a set V and two binary operations $+ : V \times V \to V$ and $F \times V \to V$ with F a field of scalars (often $V = \mathbf{R}$ or $\mathbf{C}$) satisfying the following list of axioms ($\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$; $a, b \in F$):

Associativity, $+$	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity, $+$	$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
Zero element, $+$	$\exists \mathbf{0} \in V\ \forall \mathbf{v} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$
Inverse element, $+$	$\forall \mathbf{v} \in V\ \exists \mathbf{w} \in V : \mathbf{v} + \mathbf{w} = \mathbf{0} \quad \mathbf{w} = -\mathbf{v}$
Distributivity 1	$a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$
Distributivity 2	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{w}$
"Associativity" 2	$a(b\mathbf{v}) = (ab)\mathbf{v}$
unit	$1\mathbf{v} = \mathbf{v}$

A vector space over F together with a multiplication $\cdot : V \times V \to F$ is an F-algebra if the following identities hold:

Left distributivity $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z}$
Right distributivity $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}$
Scalar identity $(a\mathbf{x})\cdot(b\mathbf{y}) = (ab)(\mathbf{x}\cdot\mathbf{y})$

Often: commutative and associative algebras.

Examples:

- Complex numbers (2D), quaternions (4D), octonions (8D)
- Function spaces $C^\infty(U, \mathbf{R})$
- Spaces of germs C_p^∞

A derivation on A is an F-linear map $D : A \to A$ satisfing the

Leibniz rule $D(fg) = (Df)g + g(Df)$.

A point derivation $D : C_p^\infty \to \mathbf{R}$ satisfies $D(fg) = Dfg(p) + f(p)Dg$.

Regular Level Set Theorem

Theorem

Let $f : N \to M$ be a C^∞ map of manifolds of dimensions $\dim M = m, \dim N = n$.
A regular level set $f^{-1}(c)$ – c a regular value – is a regular submanifold of N of dimension $n - m$.

Proof.

relies on the inverse function theorem. □

$O(n) \subset Gl(n, \mathbf{R})$ as level set

Theorem

Consider the map $f : Gl(n, \mathbf{R}) \to Gl(n, \mathbf{R})$, $f(A) = A^T A$. Then the differential f_* has constant rank.

Proof.

To $A, B \in G = Gl(n, \mathbf{R})$ associate $C = A^{-1}B$. Then $B = AC = r_C(A)$.

$$\begin{array}{ccc} A \in G & \xrightarrow{\quad f \quad} & A^T A \in G \\ \downarrow r_C & & \downarrow l_{C^T} \circ r_C \\ AC = B \in G & \xrightarrow{\ f\ } & B^T B = C^T A^T A C \in G \end{array}$$

The maps r_C and l_{C^T} are diffeomorphisms $\Rightarrow$ $(r_C)_{*,A}, (l_{C^T} \circ r_C)_{*,A^TA}$ are linear isomorphisms $\Rightarrow$ $f_{*,B} = (l_{C^T} \circ r_C)_{*,A^TA} \circ f_{*,A} \circ (r_C)^{-1}_{*,A}$ and $f_{*,A}$ have the same rank. □

Constant rank theorem for Euclidean spaces

Theorem

If $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ has constant rank k in a neighbourhood of a point $p \in U$.
Then there exists diffeomorphisms G of a neighbourhood $U' \subset U$ of p and F of a neighbourhood $V' \subset \mathbf{R}^m$ of $f(p)$ such that

$$\begin{array}{ccc} U' \subset \mathbf{R}^n & \xrightarrow{f} & V' \subset \mathbf{R}^m \\ \downarrow G & & \downarrow F \\ U'' \subset \mathbf{R}^n & \xrightarrow{F\circ f\circ G^{-1}} & V'' \subset \mathbf{R}^m \end{array}$$

such that

$$(F \circ f \circ G^{-1})(r_1, \cdots, r_n) = (r_1, \cdots, r_k, 0, \cdots 0).$$

Theorem

Let V be an open subset of $\mathbf{R}^n$ and $f : V \to \mathbf{R}^n$ a C^∞-function. For each $\mathbf{p}_0 \in V$:

1. *the system of differential equations $\mathbf{y}' = f(\mathbf{y})$ has a unique maximal smooth integral curve $\mathbf{y} : (a(p_0), b(p_0)) \to V$ with $\mathbf{y}(0) = \mathbf{p}_0$.*
2. *there is a neighbourhood $\mathbf{p}_0 \in W \subseteq V$, a number $\varepsilon > 0$, and a C^∞-function $\mathbf{y} : (-\varepsilon, \varepsilon) \times W \to V$ such that*

$$\frac{\partial \mathbf{y}}{\partial t}(t, \mathbf{q}) = f(\mathbf{y}(t, \mathbf{q})), \ \mathbf{y}(0, \mathbf{q}) = \mathbf{q}$$

for all $(t, \mathbf{q}) \in (-\varepsilon, \varepsilon) \times W$.