Algebraic Topology and Concurrency

Trace Spaces and their Applications

Martin Raussen

Department of Mathematical Sciences
Aalborg University, Denmark

ICAM 6 – Baia Mare, Romania

20.9.2008

Outline

Outline:

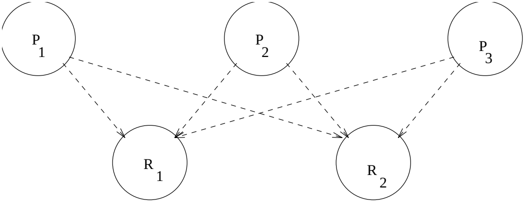
1. Motivations, mainly from Concurrency Theory (Comp.ci.)
2. Directed topology: Algebraic topology with a twist
3. Trace Spaces and their properties
4. A categorical framework (with examples and applications)

Main Collaborators:

- Lisbeth Fajstrup (Aalborg), Éric Goubault, Emmanuel Haucourt (CEA, France)

Conference: Algebraic Topological Methods in Computer Science, July 2008, Paris

Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

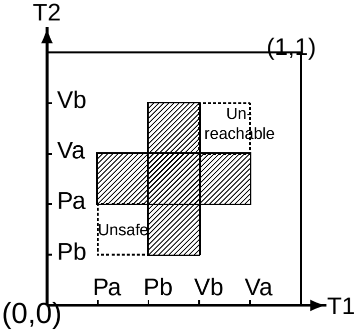
Semaphores!

Semantics: A processor has to lock a resource and relinquish the lock later on!

Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

Schedules in "progress graphs"

The Swiss flag example



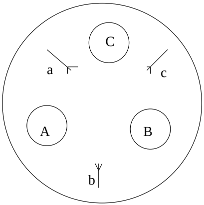
PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

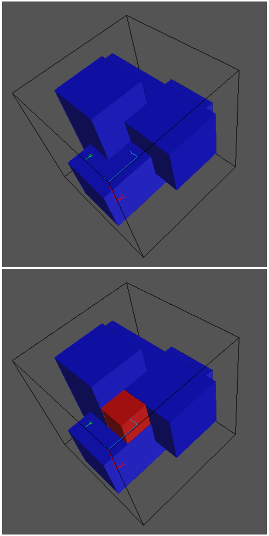
Deadlocks, unsafe and unreachable regions may occur.

Higher dimensional automata (HDA) 1

Example: Dining philosophers; dimension 3 and beyond



A=Pa.Pb.Va.Vb
B=Pb.Pc.Vb.Vc
C=Pc.Pa.Vc.Va

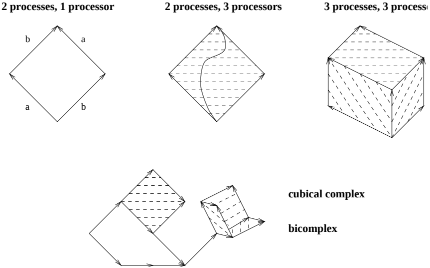


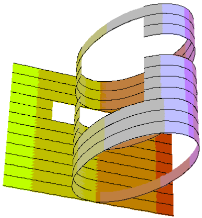
Higher dimensional complex with a forbidden region consisting of isothetic hypercubes and an unsafe region.

Higher dimensional automata (HDA) 2

seen as (geometric realizations of) pre-cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...





Squares/cubes/hypercubes are filled in iff actions on boundary are independent.

Higher dimensional automata are pre-cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps
- additionally: preferred directions – not all paths allowable.

Discrete versus continuous models

How to handle the state-space explosion problem?

Discrete models for concurrency (transition graph models) suffer a severe problem if the number of processors and/or the length of programs grows: The number of states (and the number of possible schedules) grows exponentially:
This is known as the state space explosion problem.
You need clever ways to find out which of the schedules yield equivalent results – e.g., to check for correctness – for general reasons. Then check only one per equivalence class.
Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!
Analogy: Continuous physics as an approximation to (discrete) quantum physics.

Top: the category of topological spaces and continuous maps.
$I = [0,1]$ the unit interval.

Definition

- A continuous map $H : X \times I \to Y$ is called a homotopy.
- Continuous maps $f, g : X \to Y$ are called homotopic to each other if there is a homotopy H with $H(x,0) = f(x), H(x,1) = g(x), x \in X$.
- $[X, Y]$ the set of homotopy classes of continuous maps from X to Y.
- Variation: pointed continuous maps $f : (X,*) \to (Y,*)$ and pointed homotopies $H : (X \times I, * \times I) \to (Y,*)$.
- Loops in Y as the special case $X = S^1$ (unit circle).
- Fundamental group $\pi_1(Y,y) = [(S^1,*),(Y,y)]$ with product arising from concatenation and inverse from reversal.

A framework for directed topology

d-spaces, M. Grandis (03)

X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$
a set of d-paths ("directed" paths $\leftrightarrow$ executions) satisfying

- { constant paths } $\subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization
  $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.
Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t,\ \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

d-maps, dihomotopy

A d-map $f : X \to Y$ is a continuous map satisfying

- $f(\vec{P}(X)) \subseteq \vec{P}(Y)$.

Let $\vec{P}(I) = \{\sigma \in I^I | \sigma$ nondecreasing reparametrization$\}$, and $\vec{I} = (I, \vec{P}(I))$. Then

- $\vec{P}(X) =$ set of d-maps from $\vec{I}$ to X.

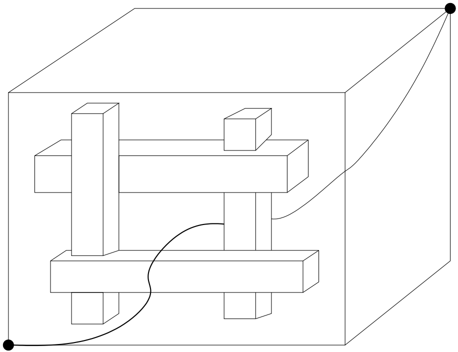
A dihomotopy $H : X \times I \to Y$ is a continuous map such that

- every H_t a d-map

i.e., a 1-parameter deformation of d-maps.

Dihomotopy is finer than homotopy with fixed endpoints

Example: Two L-shaped wedges as the forbidden region

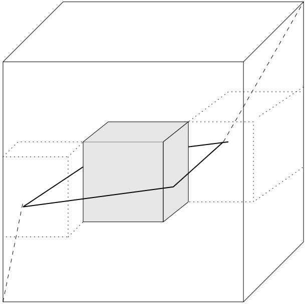


All dipaths from minimum to maximum are homotopic.
A dipath through the “hole” is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

Ordinary topology:
Path space = loop space (within each path component).
A loop space is an H-space with concatenation, inversion, cancellation.



"Birth and death" of
d-homotopy classes

Directed topology:
Loops do not tell much; concatenation ok, cancellation not!
Replace group structure by category structures!

X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$ ("same oriented trace").

Theorem

(Fahrenberg-R., 07): Reparametrization equivalence is an equivalence relation (transitivity!).

$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative.
A d-map $f : X \to Y$ induces a functor $\vec{T}(f) : \vec{T}(X) \to \vec{T}(Y)$.

The two main objectives

- Investigation/calculation of the homotopy type of trace spaces $\vec{T}(X)(x,y)$ for relevant d-spaces X
- Investigation of topology change under variation of end points:

$$\vec{T}(X)(x',y) \overset{\sigma_{x'x}^*}{\leftarrow} \vec{T}(X)(x,y) \overset{\sigma_{yy'*}}{\longrightarrow} \vec{T}(X)(x,y')$$

  Categorical organization, leading to components of end points

Application: Enough to check one d-path among all paths through the same components!

Topology of trace spaces for a pre-cubical complex X

l^1 "arc length" parametrization: on each cube, arc length is the l^1-distance of end-points. Additive continuation $\rightsquigarrow$ Subspace of arc-length parametrized d-paths $\vec{P}_n(X) \subset \vec{P}(X)$. D-homotopic paths in $\vec{P}_n(X)(x,y)$ have the same arc length! The spaces $\vec{P}_n(X)$ and $\vec{T}(X)$ are homeomorphic, $\vec{P}(X)$ is homotopy equivalent to both.

Theorem

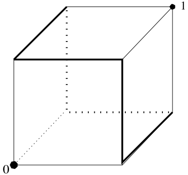
X a pre-cubical set; $x, y \in X$. Then $\vec{T}(X)(x,y)$

- *is metrizable, locally contractible and locally compact*[1].
- *has the homotopy type of a CW-complex. (using Milnor)*

First examples

I^n the unit cube, ∂I^n its boundary.

- $\vec{T}(I^n; \mathbf{x}, \mathbf{y})$ is contractible for all $x \preceq y \in I^n$;
- $\vec{T}(\partial I^n; \mathbf{0}, \mathbf{1})$ is homotopy equivalent to S^{n-2}.



[1] MR, Trace spaces in a pre-cubical complex, Draft

Aim: Decomposition of trace spaces

Method: Investigation of concatenation maps

Let $L \subset X$ denote a (properly chosen) subspace.
Investigate the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$

onto? fibres? Topology of the pieces?

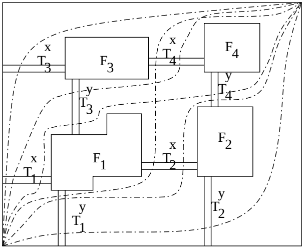
Generalization: $L_1, \cdots, L_k$ a sequence of (properly chosen) subspaces. Investigate the concatenation map on
$\vec{T}(X)(x_0, L_1) \times_{L_1} \cdots \times_{L_j} \vec{T}(X)(L_j, L_{j+1}) \times_{L_{j+1}} \cdots \times_{L_k} \vec{T}(X)(L_n, x_1).$

onto? fibres? Topology of the pieces?

Trace spaces and sequences of mutually reachable points

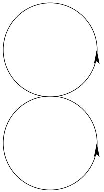
Reachability. For a given collection $\mathcal{L}$ of finitely many disjoint subsets in X that is unavoidable from x_0 to x_1, let $R^{\mathcal{L}}(L_i, L_j) = \{(x_i, x_j) \in L_i \times L_j \mid \vec{P}^{\mathcal{L}}(x_i, x_j) \neq \emptyset\} \subset X \times X$.

Theorem. *If for all $i, j, (x_i, x_j) \in R^{\mathcal{L}}(L_i, L_j)$ the trace spaces $\vec{T}^{\mathcal{L}}(X)(x_i, x_j)$ are contractible and locally contractible, then $\vec{T}(X)(x_0, x_1)$ is homotopy equivalent to the disjoint union over all $\mathcal{L}$-admissible sequences $(0, i_1, \ldots, i_n, 1)$ of spaces*
$R^{\mathcal{L}}(x_0, L_{i_1}) \times_{L_{i_1}} \cdots \times_{L_{i_j}} R^{\mathcal{L}}(L_{i_j}, L_{i_{j+1}}) \times_{L_{i_{j+1}}} \cdots \times_{L_{i_n}} R^{\mathcal{L}}(L_{i_n}, x_1) \subset X^{n+1}$.

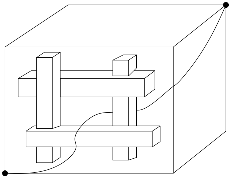


The latter space consists of sequences of mutually reachable points in the given layers.

Examples

1. 

A wedge of two directed circles $X = \vec{S}^1 \vee_{x_0} \vec{S}^1$:
$\vec{T}(X)(x_0, x_0) \simeq \{1, 2\}^*$.
(Choose $L_i = \{x_i\}, i = 1, 2$ with $x_i \neq x_0$ on the two branches).

2. 

$Y =$ cube with two wedges deleted:
$\vec{T}(Y)(\mathbf{0}, \mathbf{1}) \simeq * \sqcup (S^1 \vee S^1)$.
(L_i two vertical cuts through the wedges; product is homotopy equivalent to torus; reachability $\rightsquigarrow$ two components, one of which is contractible, the other a thickening of $S^1 \vee S^1 \subset S^1 \times S^1$.)

Piecewise linear traces

Let X denote the geometric realization of a finite pre-cubical complex ($\square$-set) M, i.e., $X = \coprod (M_n \times \vec{I}^n)_{/\simeq}$.
X consists of "cells" e_α homeomorphic to I^{n_α}. A cell is called maximal if it is not in the image of a boundary map $\partial^\pm$.
The d-path structure $\vec{P}(X)$ is inherited from the $\vec{P}(\vec{I}^n)$ by "pasting".

Definition

$p \in \vec{P}(X)$ is called PL if: $p(t) \in e_\alpha$ for $t \in J \subseteq I \Rightarrow p_{|J}$ linear[2].
$\vec{P}_{PL}(X), \vec{T}_{PL}(X)$: subspaces of linear d-paths and traces.

Theorem

For all $x_0, x_1 \in X$, the inclusion $\vec{T}_{PL}(X)(x_0, x_1) \hookrightarrow \vec{T}(X)(x_0, x_1)$ is a homotopy equivalence.

[2] and close-up on boundaries

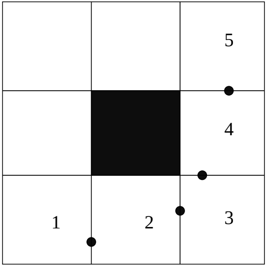
Definition

A maximal cube path in a pre-cubical set is a sequence $(e_{\alpha_1}, \ldots, e_{\alpha_k})$ of maximal cells such that $\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}} \neq \emptyset$.

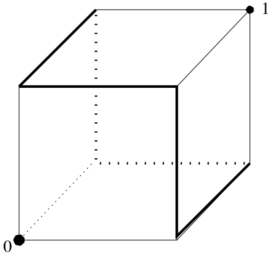
The *PL*-traces within a given maximal cube path $(e_{\alpha_1}, \ldots, e_{\alpha_k})$ correspond to sequences in $\{(y_1, \ldots, y_{k-1}) \in \prod_{i=1}^{k-1}(\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}}) \subset X^k \mid \vec{P}(e_{\alpha_i})(y_{i-1}, y_i) \neq \emptyset, 1 < i < k\}$.
This set carries a natural structure as a product of simplices $\prod \Delta^{j_k}$.

Subsimplices and their products: Some coordinates of d-paths are minimal, maximal or fixed within one or several cells.
The space $\vec{T}_{PL}(X)$ of all PL-d-paths in X is the result of pasting of these products of simplices. It carries thus the structure of a prodsimplicial complex $\rightsquigarrow$ possibilities for inductive calculations.

Simple examples

1\. 

Two maximal cube paths from **0** to **1**, each of them contributing $\Delta^2 \times \Delta^2$. Empty intersection. $\vec{T}_{PL}(X)(\mathbf{0},\mathbf{1}) \simeq (\Delta^2 \times \Delta^2) \sqcup (\Delta^2 \times \Delta^2)$.

2\. 

$X = \partial \vec{I}^n$. Maximal cube paths from **0** to **1** have length 2. Every PL-d-path is determined by an element of $\partial_{\pm} \vec{I}^n \simeq S^{n-2}$.

Future work

on the algebraic topology of trace spaces

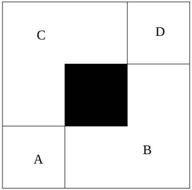
- Is there an automatic way to place consecutive “diagonal cut” layers in complexes corresponding to PV-programs that allow to compare path spaces to subspaces of the products of these layers?
- PL-d-paths come in “rounds” corresponding to the sums of dimensions of the cells they enter. This gives hope for inductive calculations (as in the work of Herlihy, Rajsbaum and others in distributed computing).
- Explore the combinatorial algebraic topology of the trace spaces
  - with fixed end points and
  - what happens under variations of end points.
- Make this analysis useful for the determination of components (extend the work of Fajstrup, Goubault, Haucourt, MR)

Categorical organization

First tool: The fundamental category

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x,y)$ depend on (x,y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
Problem: This "compression" works only for loopfree categories (d-spaces)

Preorder categories

Getting organised with indexing categories

A d-space structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: (end point) pairs $(x,y), x \preceq y$
- Morphisms:
  $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$x' \rightrightarrows x \xrightarrow{\preceq} y \rightrightarrows y'$$

- Composition: by pairwise contra-, resp. covariant concatenation.

A d-map $f : X \to Y$ induces a functor $\vec{D}(f) : \vec{D}(X) \to \vec{D}(Y)$.

The trace space functor

Preorder categories organise the trace spaces

The preorder category organises X via the
trace space functor $\vec{T}^X : \vec{D}(X) \to Top$

- $\vec{T}^X(x,y) := \vec{T}(X)(x,y)$
- $\vec{T}^X(\sigma_x, \sigma_y):$

$$\begin{array}{ccc} \vec{T}(X)(x,y) & \longrightarrow & \vec{T}(X)(x',y') \\ [\sigma] & \longmapsto & [\sigma_x * \sigma * \sigma_y] \end{array}$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms
$\vec{D}_\pi(X)((x,y),(x',y')) := \vec{\pi}_1(X)(x',x) \times \vec{\pi}_1(X)(y,y')$
and trace space functor $\vec{T}^X_\pi : \vec{D}_\pi(X) \to Ho-Top$ (with homotopy classes as morphisms).

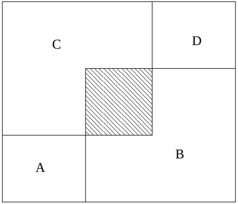
Sensitivity with respect to variations of end points

Questions from a persistence point of view

- How much does (the homotopy type of) $\vec{T}^X(x,y)$ depend on (small) changes of x,y?
- Which concatenation maps $\vec{T}^X(\sigma_x,\sigma_y): \vec{T}^X(x,y) \to \vec{T}^X(x',y'),\ [\sigma] \mapsto [\sigma_x * \sigma * \sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there "components" with (homotopically/homologically) stable dipath spaces (between them)? Are there borders ("walls") at which changes occur?

Examples of component categories

Standard example



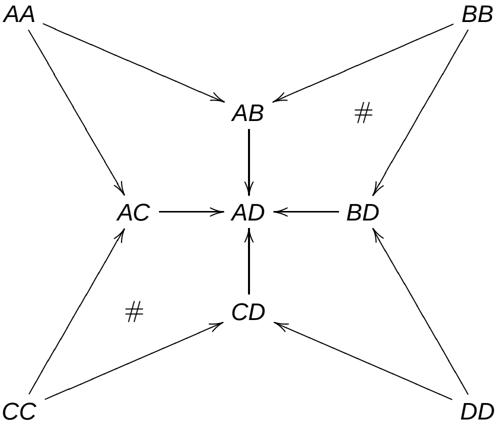


Figure: Standard example ↑ and component category →

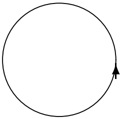
Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD$.

#: diagram commutes.

Examples of component categories

Oriented circle – with loops!

$X \quad = \quad \vec{S}^1$



$\mathcal{C}: \Delta \underset{b}{\overset{a}{\rightleftarrows}} \bar{\Delta}$

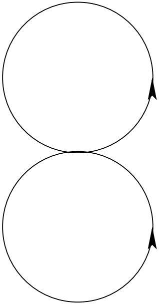
Δ the diagonal, $\bar{\Delta}$ its complement. $\mathcal{C}$ is the free category generated by a, b.

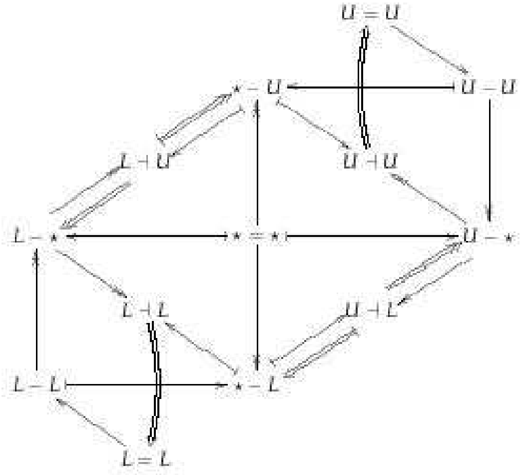
oriented circle

- Remark that the components are no longer products!
- It is essential in order to get a discrete component category to use an indexing category taking care of pairs (source, target).

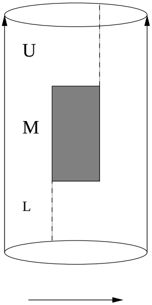
The component category of a wedge of two oriented circles

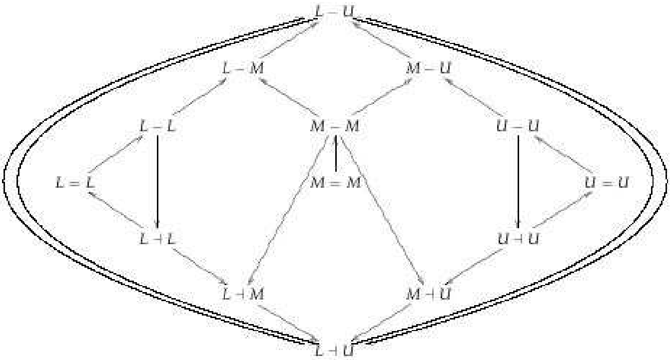
$X = \vec{S}^1 \vee \vec{S}^1$





The component category of an oriented cylinder with a deleted rectangle





Concluding remarks

- Component categories contain the essential information given by (algebraic topological invariants of) d-path spaces
- Compression via component categories is an antidote to the state space explosion problem
- Some of the ideas (for the fundamental category) are implemented and have been tested for huge industrial software from EDF (Éric Goubault & Co., CEA)
- Much more theoretical and practical work remains to be done!

Thanks for your attention!
Questions? Comments?