Spaces of executions as simplicial complexes

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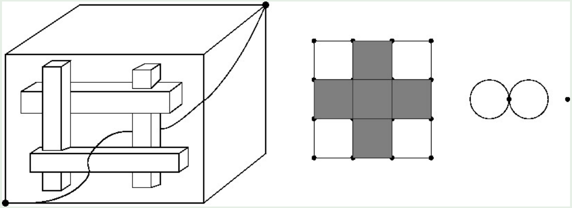
- **Homotopy type** of path space described by a matrix poset category and realized by a prodsimplicial complex
- **Algorithmics:** Detecting dead and alive subcomplexes/matrices
- **Outlook:** How to handle general HDA – with directed loops
- **Case:** Directed loops on a punctured torus (joint with L. Fajstrup (Aalborg) K. Ziemiański, (Warsaw))

Intro: State space, directed paths and trace space

Problem: How are they related?

Example 1: State space and trace space for a semaphore HDA



State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

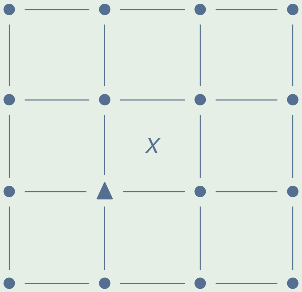
Path space model contained in torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

Analogy in standard algebraic topology

Relation between space X and loop space ΩX.

Intro: State space and trace space
with loops

Example 2: Punctured torus



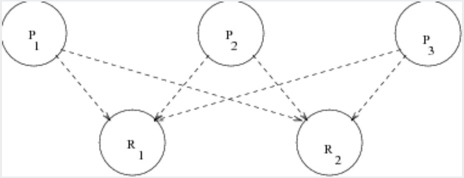
State space: Punctured torus X and branch point ▲:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$.

Question: Path space for a punctured torus in higher dimensions?
Joint work with L. Fajstrup and K. Ziemiański.

Mutual exclusion

occurs, when *n* processes P_i compete for *m* resources R_j.



Only *k* processes can be served at any given time.

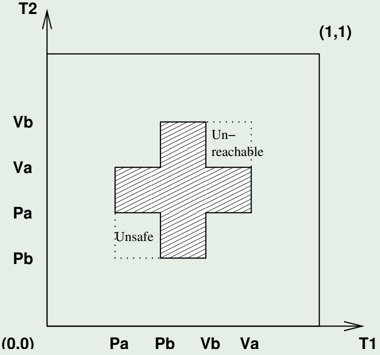
Semaphores

Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

P: probeer; *V*: verhoog

Semaphores: The Swiss flag example



PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:

- Hole = isothetic hyperrectangle $R^i =]a^i_1, b^i_1[\times \cdots \times]a^i_n, b^i_n[\subset I^n, 1 \leq i \leq l$: with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
- State space $X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^{l} R^i$ X inherits a partial order from $\vec{I}^n$. d-paths are order preserving.

More general concurrent programs ⇝ HDA

Higher Dimensional Automata (HDA, V. Pratt; 1990):

- Cubical complexes: like simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.
- d-paths are order preserving.

Definition

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a, b) = \{p : \vec{I} \to X | \, p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a, b) = \vec{P}(X)(a, b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a, b) \simeq \vec{T}(X)(a, b)$.
- A dihomotopy in $\vec{P}(X)(a, b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a, b), \; t \in I$; ie a path in $\vec{P}(X)(a, b)$.

Aim:

Description of the homotopy type of $\vec{P}(X)(a, b)$ as explicit finite dimensional (prod-)simplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

Tool: Subspaces of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

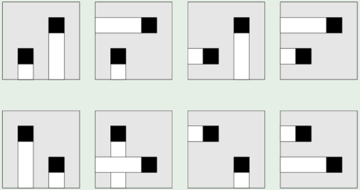
$X = \vec{I}^n \setminus F, F = \bigcup_{i=1}^{l} R^i; R^i = [\mathbf{a}^i, \mathbf{b}^i]; \mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

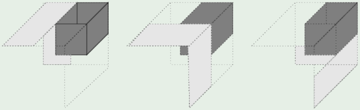
1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – direction j restricted at hole i
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ – Which directions are restricted at which hole?

Examples: two holes in 2D – one hole in 3D (dark)

$M =$
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



$M =$
[100] [010] [001]



Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices
$M_{l,n}^{R,*}$ no row vector is the zero vector – every hole obstructed in at least one direction

A cover by contractible subspaces

Theorem

1. $$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,*}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible.* *Which is which?*

Proof.

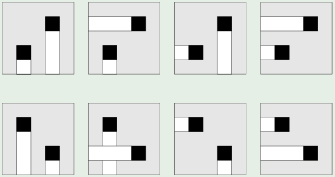
Subspaces X_M, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

Combinatorics
poset category
$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^{R,*} \subseteq M_{l,n}$
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ "alive"

Topology:
prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^{l}$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq$
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ – one simplex Δ_{m_i} for every hole

$$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$$

Examples of path spaces



$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$

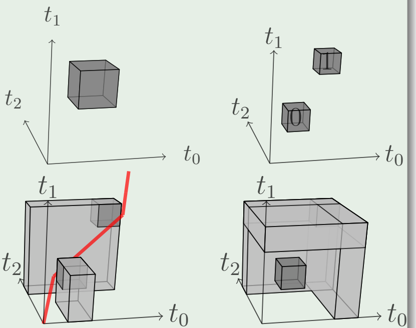
- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

$\supset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

Further examples

State spaces, "alive" matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{1,n}^{R,*} \setminus \{[1,\dots,1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2. $X = \vec{I}^n \setminus (\vec{J}_0^n \cup \vec{J}_1^n)$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{2,n}^{R,*} \setminus$ matrices with a $[1,\dots,1]$-row.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq S^{n-2} \times S^{n-2}$.



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

alive dead

Homotopy equivalence between path space $\vec{P}(X)(\mathbf{0},\mathbf{1})$ and prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem (A variant of the nerve lemma)

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  hocolim $\mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  hocolim $\mathcal{D} \simeq$ hocolim $\mathcal{T}^* \simeq$ hocolim $\mathcal{T} \simeq$ hocolim $\mathcal{E}$.
- Projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

□

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL at CEA/LIX-lab.: Goubault, Haucourt, Mimram
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek's CrHom etc) to calculate the homology groups of these chain complexes even for quite big complexes: M. Juda (Krakow).
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
  For path-components alone, there are fast "discrete" methods, that also yield representatives in each path component (ALCOOL).

Open problem: Huge complexes – complexity

Huge prodsimplicial complexes

l obstructions, n processors:
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ is a subcomplex of $(\partial\Delta^{n-1})^l$:
potentially a huge high-dimensional complex.

Possible antidotes

- Smaller models? Make use of partial order among the obstructions R^i, and in particular the inherited partial order among their extensions R^i_j with respect to $\subseteq$.
- Work in progress: yields simplicial complex of far smaller dimension!

Components?!

- So far: $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ - fixed end points.
- Now: Variation of $\vec{T}(X)(\mathbf{a}, \mathbf{b})$ of start and end point, giving rise to filtrations.
- At which thresholds do homotopy types change?
- How to cut up $X \times X$ into components so that the homotopy type of trace spaces with end point pair in a component is invariant?
- Birth and death of homology classes?
- Compare with multidimensional persistence (Carlsson, Zomorodian).

Case: d-paths on a punctured torus

Punctured torus and n-space

n-torus $T^n = \mathbf{R}^n/_{\mathbf{Z}^n}$.

forbidden region $F^n = ([\frac{1}{4}, \frac{3}{4}]^n + \mathbf{Z}^n)/_{\mathbf{Z}^n} \subset T^n$.

punctured torus $Y^n = T^n \setminus F^n$

punctured n-space [a] $\tilde{Y}^n = \mathbf{R}^n \setminus ([\frac{1}{4}, \frac{3}{4}]^n + \mathbf{Z}^n)$

with d-paths from quotient map $\mathbf{R}^n \downarrow T^n$.

[a]universal cover

Aim: Describe the homotopy type of $\vec{P}(Y) = \vec{P}(Y)(\mathbf{0}, \mathbf{0})$

$\vec{P}(Y) \hookrightarrow \Omega Y(\mathbf{0}, \mathbf{0}) \rightsquigarrow$ disjoint union $\vec{P}(Y) = \bigsqcup_{\mathbf{k} \geq \mathbf{0}} \vec{P}(\mathbf{k})(Y)$
with multiindex = multidegree $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{Z}_+^n, k_i \geq 0$.
$\vec{P}(\mathbf{k})(Y) \cong \vec{P}(\tilde{Y}^n)(\mathbf{0}, \mathbf{k}) =: Z(\mathbf{k})$.

Path spaces as colimits

Category $\mathcal{J}(n)$

Poset category of proper non-empty subsets of $[1:n]$ with inclusions as morphisms.
Via characteristic functions isomorphic to the category of non-identical bit sequences of length n: $\varepsilon = (\varepsilon_1, \dots \varepsilon_n) \in \mathcal{J}(n)$.
$B\mathcal{J}(n) \cong \partial\Delta^{n-1} \cong S^{n-2}$.

Definition

$U_\varepsilon(\mathbf{k}) := \{\mathbf{x} \in \mathbf{R}^n | \ \varepsilon_j = 1 \Rightarrow x_j \leq k_j - \frac{3}{4} \text{ or } \exists i : x_i \geq k_i - \frac{1}{4}\}$
$Z_\varepsilon(\mathbf{k}) := \vec{P}(U_\varepsilon(\mathbf{k}))(\mathbf{0}, \mathbf{k})$.

Lemma

$Z_\varepsilon(\mathbf{k}) \simeq Z(\mathbf{k} - \varepsilon)$.

Theorem

$Z(\mathbf{k}) = \operatorname{colim}_{\varepsilon \in \mathcal{J}(n)} Z_\varepsilon(\mathbf{k}) \simeq \operatorname{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z_\varepsilon(\mathbf{k}) \simeq \operatorname{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon)$.

An equivalent homotopy colimit construction

Inductive homotopy colimites

Using the category $\mathcal{J}(n)$ construct for $\mathbf{k} \in \mathbf{Z}^n, \mathbf{k} \geq \mathbf{0}$:

- $X(\mathbf{k}) = *$ if $\prod_1^n k_i = 0$;
- $X(\mathbf{k}) = \text{hocolim}_{\varepsilon \in \mathcal{J}(n)} X(\mathbf{k} - \varepsilon)$.

By construction $\mathbf{k} \leq \mathbf{l} \Rightarrow X(\mathbf{k}) \subseteq X(\mathbf{l})$; $X(\mathbf{1}) \cong \partial\Delta^{n-1}$.

Inductive homotopy equivalences

$q(\mathbf{k}) : Z(\mathbf{k}) \to X(\mathbf{k})$:

- $\prod_1^n k_i = 0 \Rightarrow Z(\mathbf{k})$ contractible, $X(\mathbf{k}) = *$
- $q(\mathbf{k}) = \text{hocolim}_{\varepsilon \in \mathcal{J}(n)} q(\mathbf{k} - \varepsilon) : Z(\mathbf{k}) \cong \text{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon) \to \text{hocolim}_{\varepsilon \in \mathcal{J}(n)} X(\mathbf{k} - \varepsilon) = X(\mathbf{k})$.

Homology and cohomology of space $Z(\mathbf{k})$ of d-paths

Definition

- $\mathbf{l} \ll \mathbf{m} \in \mathbf{Z}_+^n \Leftrightarrow l_j < m_j, 1 \leq j \leq n$.
- $\mathcal{O}^n = \{(\mathbf{l}, \mathbf{m}) | \; \mathbf{l} \ll \mathbf{m} \text{ or } \mathbf{m} \ll \mathbf{l}\} \subset \mathbf{Z}_+^n \times \mathbf{Z}_+^n$.
- $\mathbf{B}(\mathbf{k}) := \mathbf{Z}_+^n(\leq \mathbf{k}) \times \mathbf{Z}_+^n(\leq \mathbf{k}) \setminus \mathcal{O}^n$.
- $\mathcal{I}(\mathbf{k}) :=< \mathbf{lm} | \; (\mathbf{l}, \mathbf{m}) \in \mathbf{B}(\mathbf{k}) > \leq \mathbf{Z}[\mathbf{Z}_+^n(\leq \mathbf{k})]$.

Theorem

For $n > 2$, $H^*(Z(\mathbf{k})) = \mathbf{Z}[\mathbf{Z}_+^n(\leq \mathbf{k})]/_{\mathcal{I}(\mathbf{k})}$.
$H_*(Z(\mathbf{k})) \cong H^*(Z(\mathbf{k}))$ as abelian groups.

Proof

Spectral sequence argument, using projectivity of the functor $H_* : \mathcal{J}(n) \to \mathbf{Ab}_*, \; \mathbf{k} \mapsto H_*(Z(\mathbf{k}))$

Interpretation via cube sequences
Betti numbers

Cube sequences

$[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^r = \mathbf{l}] \in A^n_{r(n-2)}(\mathbf{l})$ - of size $\mathbf{l} \in \mathbf{Z}^n_+$, length r and degree $r(n-2)$.
$A^n_*(*)$ the free abelian group generated by all cube sequences.
$A^n_*(\leq \mathbf{k}) := \bigoplus_{\mathbf{l} \leq \mathbf{k}} A^n_*(\mathbf{l})$.
$H_{r(n-2)}(Z(\mathbf{k})) \cong A^n_{r(n-2)}(\leq \mathbf{k})$ – generated by cube sequences of length r and size $\leq \mathbf{k}$.

Betti numbers of $Z(\mathbf{k})$

Theorem

$$n = 2: \; \beta_0 = \binom{k_1+k_2}{k_1}; \beta_j = 0, \; j > 0;$$
$$n > 2: \; \beta_0 = 1, \; \beta_{i(n-2)} = \prod_1^n \binom{k_j}{i}, \; \beta_j = 0 \textit{ else}.$$

Corollary

1. *Small homological dimension of* $Z(\mathbf{k})$: $(\min_j k_j)(n-2)$.
2. *Duality: For* $\mathbf{k} = (k, \ldots, k)$, $\beta_i(Z(\mathbf{k})) = \beta_{k(n-2)-i}(Z(\mathbf{k}))$. *Why?*

To conclude

Conclusions and challenges

- From a (rather compact) state space model (shape of data) to a finite dimensional trace space model (represent shape).
- Calculations of invariants (Betti numbers) of path space possible for state spaces of a moderate size (measuring shape).
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors); still: curse of dimensionality.
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  Connections to the work of Herlihy and Rajsbaum protocol complex etc
- Challenge: Morphisms between HDA $\rightsquigarrow$ d-maps between cubical state spaces $\rightsquigarrow$ functorial maps between trace spaces. Properties? Equivalences?

Want to know more?

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