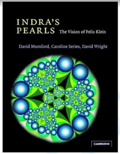


Highly complex: Möbius transformations, hyperbolic tessellations and pearl fractals

Martin Raussen

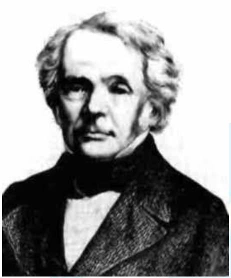
Department of mathematical sciences
Aalborg University

Cergy-Pontoise 26.5.2011

Möbius transformations

Definition

- Möbius transformation:
  a rational function $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of the form
  $f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbf{C}$, $ad - bc \neq 0$.
- $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.
- $f(-d/c) = \infty, f(\infty) = a/c$.



August Ferdinand Möbius
1790 – 1868

Examples of Möbius transformations

Imagine them on the Riemann sphere

Translation $z \mapsto z + b$

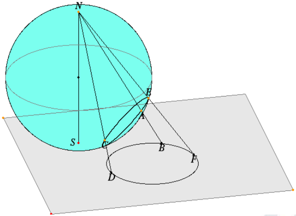
Rotation $z \mapsto (cos\theta + i \sin\theta) \cdot z$

Zoom $z \mapsto az, a \in \mathbf{R}, a > 0$

Circle inversion $z \mapsto 1/z$

Stereographic projection allows to identify the unit sphere S^2 with $\overline{\mathbf{C}}$.
How do these transformations look like on the sphere?

Have a look!



The algebra of Möbius transformations

2×2-matrices

- $GL(2, \mathbf{C})$: the group af all invertible 2×2-matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with complex coefficients; invertible: $\det(A) = ad - bc \neq 0$.
- $A \in GL(2, \mathbf{C})$ corresponds to the MT $z \mapsto \frac{az+b}{cz+d}$.
- Multiplication of matrices corresponds to composition of transformations.
- The Möbius transformation given by a matrix A has an inverse Möbius transformation given by A^{-1}.
- The matrices A og $rA, r \neq 0$, describe the *same* MT.
- Hence the group of Möbius transformations is isomorphic to the projective group $PGL(2, \mathbf{C}) = GL(2, \mathbf{C})_{/\mathbf{C}^*}$ – a $8 - 2 = 6-$ dimensional Lie group: 6 real degrees of freedom.

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Theorem

1. *Every Möbius transformation is a composition of translations, rotations, zooms (dilations) and inversions.*
2. *A Möbius transformation is* **conformal** *(angle preserving).*
3. *A Möbiustransformation maps* **circles into circles** *(straight line = circle through ∞).*
4. *Given two sets of 3 distinct points P_1, P_2, P_3 and Q_1, Q_2, Q_3 in $\bar{\mathbf{C}}$. There is* **one MT** *f with* $f(P_i) = Q_i$.

Proof.

(1) $\frac{az+b}{cz+d} = \frac{a}{c} + \frac{(bc-ad)/c^2}{z+d/c}$.

(4) To map (P_1, P_2, P_3) to $(0, 1, \infty)$, use $f_P(z) = \frac{(z-P_1)(P_2-P_3)}{(z-P_3)(P_2-P_1)}$

$f_Q : (Q_1, Q_2, Q_3) \mapsto (0, 1, \infty)$.

$f := (f_Q)^{-1} \circ f_P$.

Uniqueness: Only *id* maps $(0, 1, \infty)$ to $(0, 1, \infty)$.

Three complex degrees of freedom! □

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The geometry of Möbius transformations 2

Conjugation and fix points

Two Möbius transformations f_1, f_2 are conjugate if there exists a Möbius transformation T (a “change of coordinates”) such that

$$f_2 = T \circ f_1 \circ T^{-1}.$$

Conjugate Möbius transformations have similar geometric properties; in particular the same number of fixed points, invariant circles etc.

A Möbius transformation ($\neq id$) has either two fix points or just one.

If a MT has two fix points, then it is conjugate to one of the form $z \mapsto az$. $z \mapsto \frac{1}{z}$?

If a MT has only one fixed point, then it is conjugate to a translation $z \mapsto z + b$.

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Geometric and algebraic classification
the trace!

A Möbius transformation can be described by a matrix A with $\det(A) = 1$ (almost uniquely). Consider the trace $Tr(A) = a + d$ of such a corresponding matrix A.

The associated Möbius transformation ($\neq id$) is

- **parabolic** (one fix point): conjugate to $z \mapsto z + b \Leftrightarrow Tr(A) = \pm 2$
- **elliptic** (invariant circles): conjugate to $z \mapsto az,\ |a| = 1 \Leftrightarrow Tr(A) \in]-2, 2[$
- **loxodromic** conjugate to $z \mapsto az,\ |a| \neq 1 \Leftrightarrow Tr(A) \notin [-2, 2]$

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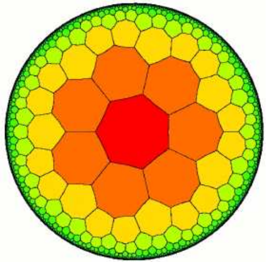
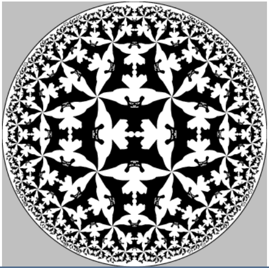
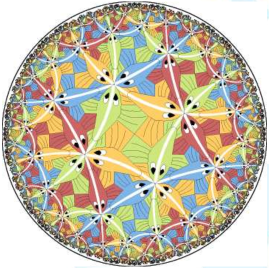
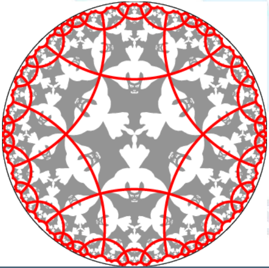
Examples
M.C. Escher (1898 – 1972)

Background: Hyperbolic geometry

Models: Eugenio Beltrami, Felix Klein, Henri Poincaré

Background for classical geometry: Euclid, based on 5 postulates.
2000 years of struggle concerning the parallel postulate: Is it independent of/ios it a consequence of the 4 others?
Gauss, Bolyai, Lobachevski, 1820 – 1830: Alternative geometries, angle sum in a triangle differs from 180^0.
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Models for hyperbolic geometry

Geodesic curves, length, angle

Poincaré's upper half plane:

$H = \{z \in \mathbf{C} | \Im z > 0\}$. Geodesic curves (lines): half lines and half circles perpendicular on the real axis.
Angles like in Euclidean geometry.
Length: line element $ds^2 = \frac{dx^2+dy^2}{y}$ – real axis has distance ∞ from interior.

Poincaré's disk: $D = \{z \in \mathbf{C} | |z| < 1\}$.
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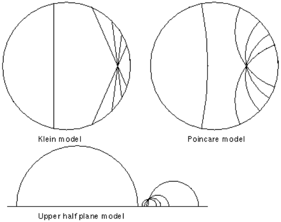
Figure 1: The same lines in three different models. Note that the parallel postulate fails here.



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Same disc. Geodesic curves = secants.
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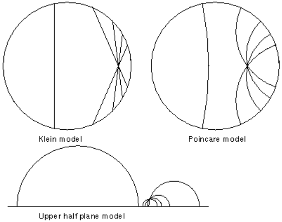

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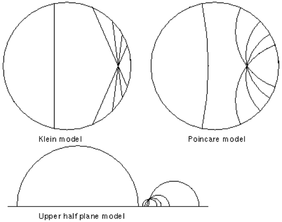

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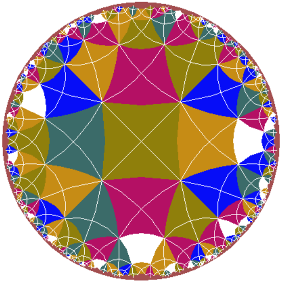
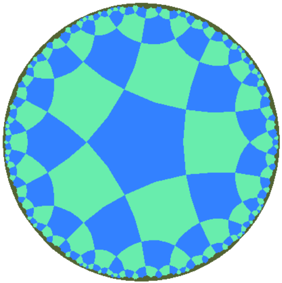
Hyperbolic tesselations

Regular tesselation in Euklidean geometry – Schläfli symbols:
Only $(n, k) = (3, 6), (4, 4), (6, 3)$ – k regular n-gons – possible.
Angle sum = $180^0 \Rightarrow \frac{1}{n} + \frac{1}{k} = \frac{1}{2}$.
in hyperbolic geometry: $\frac{1}{n} + \frac{1}{k} < \frac{1}{2}$: Infinitely many possibilities!





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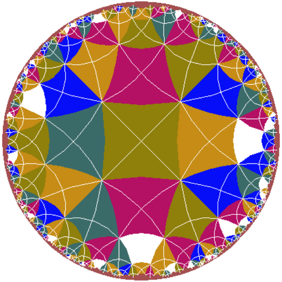

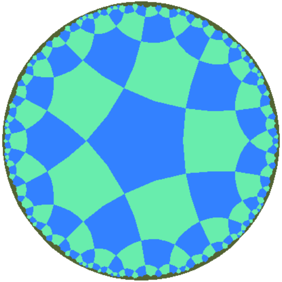

Pattern preserving transformations form a discrete subgroup or the group of all Möbius transformations.
Do it yourself! 2

Schottky groups

Discrete subgroups within Möbius transformations

Given two disjoint circles C_1, D_1 in $\mathbf{C}$.
There is a Möbius transformation A mapping the outside/inside of C_1 into the inside/ouside of C_2. What does $a = A^{-1}$?
Correspondingly: two disjoint circles C_2, D_2 in $\mathbf{C}$, disjoint with C_1, D_1. Möbius transformations B, b.
The subgroup $< A, B >$ generated by A, B consists of all "words" in the alphabet A,a,B,b (only relations: $Aa = aA = e = Bb = bB$).
Examples:
$A, a, B, b, A^2, AB, Ab, a^2, aB, ab, BA, Ba, B^2, bA, ba, b^2, A^3, A^2B, ABa, \ldots$

How do the transformations in this (Schottky)-subgroup act on $\bar{\mathbf{C}}$?

Friedrich Schottky 1851 – 1935

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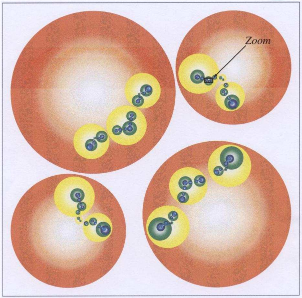
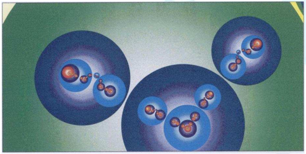
From Schottky group to fractal

- One step: Apply (one of) the operations A, a, B, b.
- Result: Three outer disks are “copied” into an inner disk.
- These “new” circles are then copied again in the next step.
- “Babushka” principle: Copy within copy within copy...$\rightsquigarrow$ a point in the limit set $\rightsquigarrow$ fractal.
- What is the shape of this limit set?





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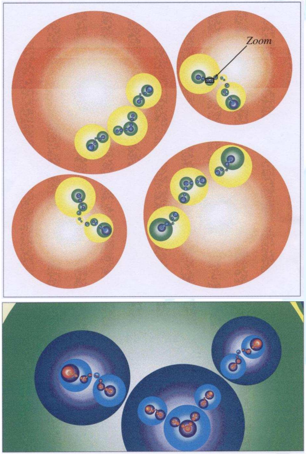

Kleinian groups, Fuchsian groups and limit sets

Background and terminology

Definition

Kleinian group: a **discrete** subgroup of Möbius transformations

Fuchsian group: a Kleinian group of Möbius transformations that preserve the upper half plane H (hyperbolic isometries, real coefficients)

Orbit: of a point $z_0 \in \mathbf{C}$ under the action of a group G: $\{g \cdot z_0 | g \in G\}$

Limit set: $\Lambda(G)$: consists of all limit points of alle orbits.

Regular set: $\Omega(G) := \overline{\mathbf{C}} \setminus \Lambda(G)$.

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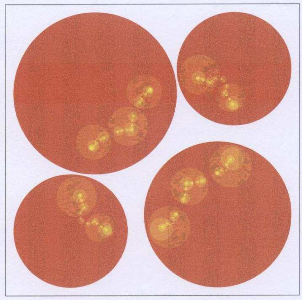
Limit sets for Schottky grpups
Properties

starting with disjoint circles:

The limit set $\Lambda(G)$ for a Schottky group G is a fractal set. It

- is totally disconnected;
- has positive Hausdorff dimension;
- has area 0 (fractal "dust").



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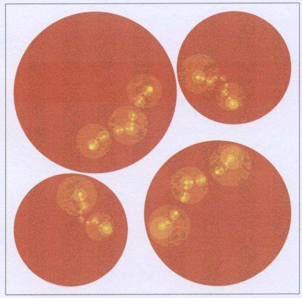

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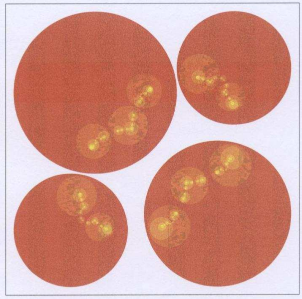
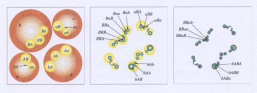


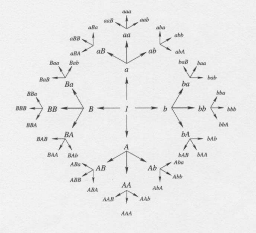
Cayley graph and limit fractal

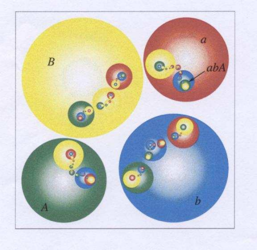
Convergence of “boundaries” in the Cayley graph

Every limit point in $\Lambda(G)$ corresponds to an infinite word in the four symbols A, a, B, b (“fractal mail addresses”).

The limit fractal $\Lambda(G)$ corresponds also to the boundary of the Cayley graph for the group G – the metric space that is the limit of the boundaries of words of limited length (Abel prize recipient M. Gromov).







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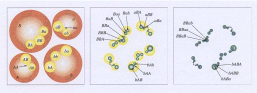
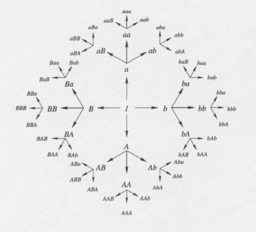
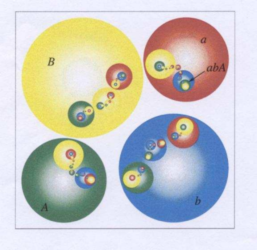


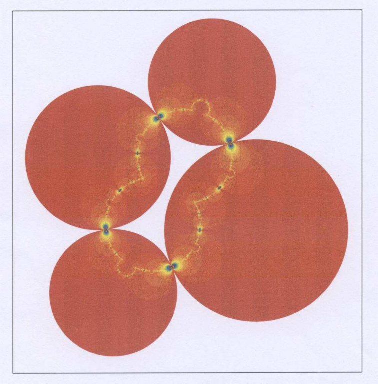




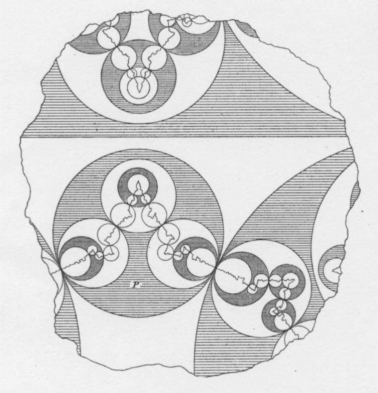
“Kissing Schottky groups” and fractal curves

For tangent circles

The dust connects up and gives rise to a fractal curve:



F. Klein and R. Fricke knew that already back in 1897 – without access to a computer!



Have a try!

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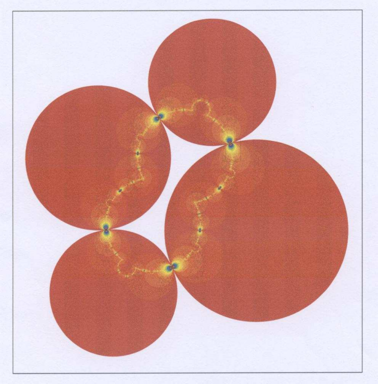

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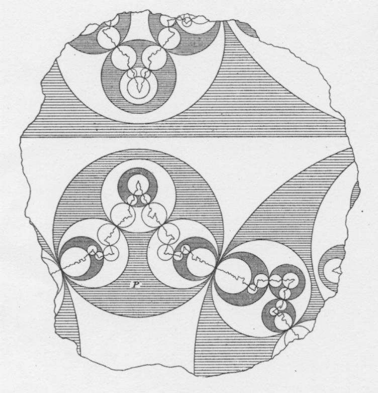

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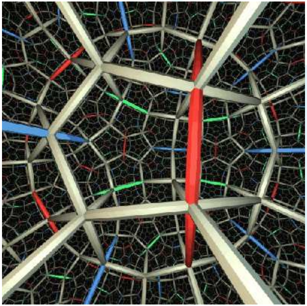
Outlook to modern research: 3D hyperbolic geometry

following Poincaré's traces

Model: 3D ball with boundary sphere S^2 (at distance ∞ from interior points).
"Planes" in this model:
Spherical caps perpendicular to the boundary.

Result: a 3D tesselation by hyperbolic polyhedra.
To be analyzed at $S^2 = \overline{\mathbf{C}}$ on which the full Möbius group $PGL(2, \mathbf{C})$ acts.
Most 3D-manifolds can be given a hyperbolic structure (Thurston, Perelman).



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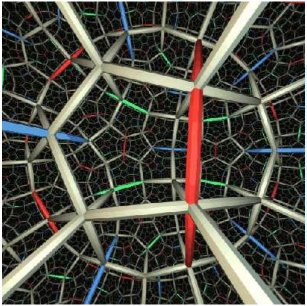

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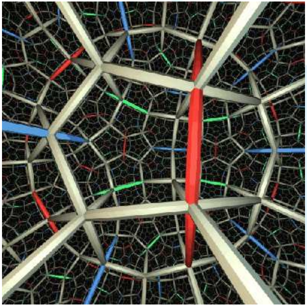
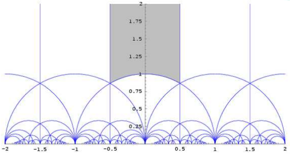


Möbius transformations and number theory
Modular forms

Modular group consists of Möbius transformations with integer coefficients: $PSL(2, \mathbf{Z})$.
Acts on the upper half plane $\mathbf{H}$.

Fundamental domains boundaries composed of circular arcs.



Modular form Meromorphic function satisfying

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z).$$

Important tool in

Analytic number theory Moonshine. Fermat-Wiles-Taylor.

References

partially web based

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Thanks!

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