Simplicial models for trace spaces

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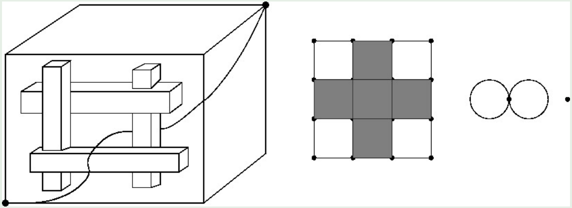
Problem: How are they related?

Example 1: State space and trace space for a semaphore HDA



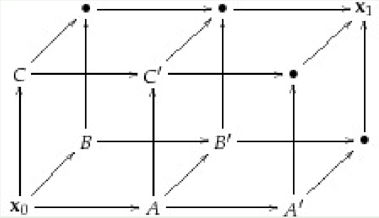
State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

Path space model contained in torus $(\partial\Delta^2)^2$ –
homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

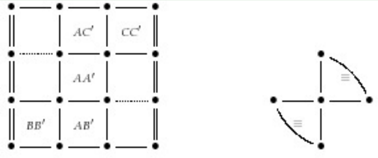
Analogy in standard algebraic topology

Relation between space X and loop space ΩX.

Example 2: State space and trace space for a non-looping pre-cubical complex

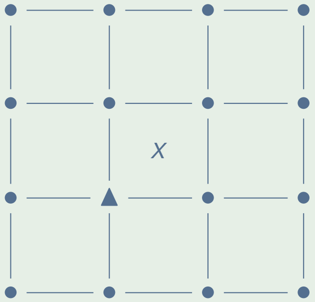


State space: Boundaries of two cubes glued together at common square $AB'C'\bullet$



Path space model: Prodsimplicial complex contained in $(\partial\Delta^2)^2 \cup \partial\Delta^2$– homotopy equivalent to $S^1 \vee S^1$

Example 3: Torus with a hole



State space with hole X:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$.

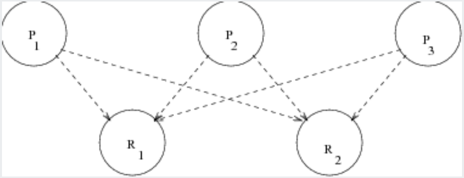
Question: Path space for a torus with hole in higher dimensions?

Motivation: Concurrency

Semaphores: A simple model for mutual exclusion

Mutual exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores

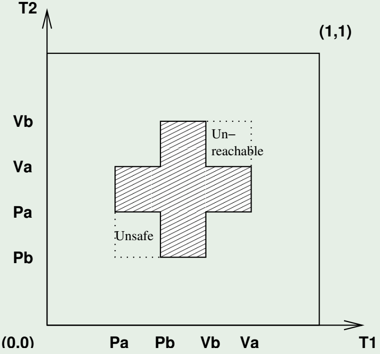
Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

P: probeer; V: verhoog

A geometric model: Schedules in "progress graphs"

Semaphores: The Swiss flag example



PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:

Hole = isothetic hyperrectangle

$R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[\subset I^n, 1 \le i \le l$:

with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.

State space $X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^l R^i$

X inherits a partial order from $\vec{I}^n$. d-paths are order preserving.

More general (PV)-programs:

- Replace $\vec{I}^n$ by a product $\Gamma_1 \times \cdots \times \Gamma_n$ of digraphs.
- Holes have then the form $p_1^i((0,1)) \times \cdots \times p_n^i((0,1))$ with $p_j^i : \vec{I} \to \Gamma_j$ a directed injective (d-)path.
- Pre-cubical complexes: like pre-simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.

Definition

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a,b) = \{p : \vec{I} \to X \mid p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
- A dihomotopy in $\vec{P}(X)(a,b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a,b),\ t \in I$; ie a path in $\vec{P}(X)(a,b)$.

Aim:

Description of the homotopy type of $\vec{P}(X)(a,b)$ as explicit finite dimensional prodsimplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

Tool: Subspaces of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

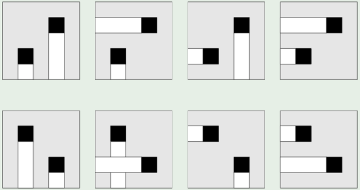
$X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^{l} R^i;\ R^i = [\mathbf{a}^i, \mathbf{b}^i];\ \mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

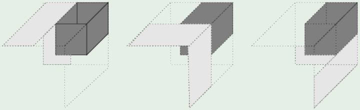
1. $X_{ij} = \{x \in X \mid x \le \mathbf{b}^i \Rightarrow x_j \le a^i_j\}$ – direction j restricted at hole i
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ –
Which directions are restricted at which hole?

Examples: 2 holes in 2D/ 1hole in 3D

$M =$
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



$M =$
[100] [010] [001]



Covers by contractible (or empty) subspaces

Bookkeeping with binary matrices

Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^{R,*}$ no row vector is the zero vector

$M_{l,n}^{R,u}$ every row vector is a unit vector

$M_{l,n}^{C,u}$ every column vector is a unit vector

A cover:

$$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,*}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

Theorem

Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible. *Which is which?*

Proof.

Subspaces X_M, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

A combinatorial model and its geometric realization

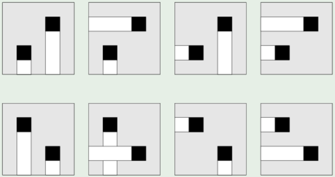
First examples

Combinatorics
poset category
$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^{R,*} \subseteq M_{l,n}$
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ "alive"

Topology:
prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq$
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ – one simplex Δ_{m_i} for every hole

$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset$.

Examples of path spaces



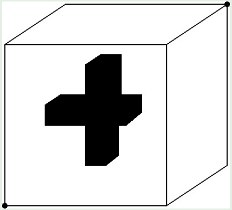
$\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$

- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

$\supset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

Further examples

State spaces, “alive” matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
2. 

1.
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{1,n}^{R,*} \setminus \{[1,\dots 1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2.
   - $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\}$.
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = \{M \in M_{l,n}^{R,*} \mid \exists N \in \mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) : M \leq N\}$
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) =$ 3 diagonal squares $\subset (\partial\Delta^2)^2 = T^2 \simeq S^1$.

Homotopy equivalence between trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ and the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem (A variant of the nerve lemma)

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  hocolim $\mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  hocolim $\mathcal{D} \cong$ hocolim $\mathcal{T}^* \cong$ hocolim $\mathcal{T} \cong$ hocolim $\mathcal{E}$.
- Projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

□

Why prodsimplicial?
rather than simplicial

- We distinguish, for every obstruction, sets $J_i \subset [1:n]$ of restrictions. A joint restriction is of product type $J_1 \times \cdots \times J_l \subset [1:n]^l$, and not an arbitrary subset of $[1:n]^l$.
- Simplicial model: a subcomplex of Δ^{n^l} – $2^{(n^l)}$ subsimplices.
- Prodsimplicial model: a subcomplex of $(\Delta^n)^l$ – $2^{(nl)}$ subsimplices.

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL:
  Progress at CEA/LIX-lab.: Goubault, Haucourt, Mimram
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek's CrHom etc) to calculate the homology groups of these chain complexes even for very big complexes: M. Juda (Krakow).
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
  For path-components alone, there are faster "discrete" methods, that also yield representatives in each path component.

Detection of dead and alive subcomplexes

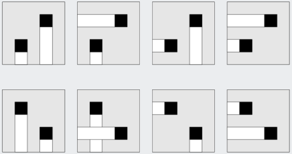
An algorithm starts with deadlocks and unsafe regions!

Allow less = forbid more!

Remove **extended** hyperrectangles

$R^i_j := [0, b^i_1[\times \cdots \times [0, b^i_{j-1}[\times]a^i_j, b^i_j[\times [0, b^i_{j+1}[\times \cdots \times [0, b^i_n[\supset R^i$.

$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j$.



Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
2. *There is a "**dead**" matrix* $N \leq M$, $N \in M^{C,u}_{l,n}$, *such that* $\bigcap_{n_{ij}=1} R^i_j \neq \emptyset$ *– giving rise to a **deadlock** unavoidable from* $\mathbf{0}$, *i.e.,* $T(X_N)(\mathbf{0},\mathbf{1}) = \emptyset$.

Dead matrices in $D(X)(\mathbf{0},\mathbf{1})$

Inequalities decide

Decisions: Inequalities

Deadlock algorithm (Fajstrup, Goubault, Raussen) $\rightsquigarrow$:

Theorem

- $N \in M_{l,n}^{C,u}$ *dead* $\Leftrightarrow$
  For all $1 \leq j \leq n$, *for all* $1 \leq k \leq n$ *such that* $\exists j' : n_{kj'} = 1$:

$$n_{ij} = 1 \Rightarrow a_j^i < b_j^k.$$

- $M \in M_{l,n}^{R,*}$ *dead* $\Leftrightarrow \exists N \in M_{l,n}^{C,u}$ *dead*, $N \leq M$.

Definition

$D(X)(\mathbf{0},\mathbf{1}) := \{P \in M_{l,n} | \exists N \in M_{l,n}^{C,u}, N$ *dead* $: N \leq P\}$.

A cube with a cube hole

- $X = \vec{I}^n \setminus \vec{J}^n$
- $D(X)(\mathbf{0},\mathbf{1}) = \{[1,\dots,1]\} = M_{1,n}^{C,u}$.

Maximal alive $\leftrightarrow$ minimal dead

Still alive – not yet dead

- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1}) \subset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ maximal alive matrices.
- Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.
- Connection: $M \in \mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1}), M \leq N$ a succesor (a single 0 replaced by a 1) $\Rightarrow N \in D(X)(\mathbf{0},\mathbf{1})$.

A cube removed from a cube

- $X = \vec{I}^n \setminus \vec{J}^n, D(X)(\mathbf{0},\mathbf{1}) = \{[1,\ldots,1]\}$;
- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$: vectors with a single 0;
- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{l,n} \setminus \{[1,\ldots,1]\}$;
- $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1}$.

Open problem: Huge complexes – complexity

- l obstructions, n processors:
  $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ is a subcomplex of $(\partial\Delta^{n-1})^l$:
  potentially a huge high-dimensional complex.
- Smaller models? Make use of partial order among the obstructions R^i, and in particular the inherited partial order among their extensions R^i_j with respect to $\subseteq$.
- Consider only saturated matrices in the sense:
  $R^{i_1}_j \subset R^{i_2}_j, m_{i_2 j} = 1 \Rightarrow m_{i_1 j} = 1$.
- Work in progress: yields simplicial complex of far smaller dimension!

Open problem: Variation of end points

Conncection to MD persistence?

- So far: $\vec{T}(X)(\mathbf{0},\mathbf{1})$ - fixed end points.
- Now: Variation of $\vec{T}(X)(\mathbf{a},\mathbf{b})$ of start and end point, giving rise to filtrations.
- At which thresholds do homotopy types change?
- Can one cut up $X \times X$ into components so that the homotopy type of trace spaces with end point pair in a component is invariant?
- Birth and death of homology classes?
- Compare with multidimensional persistence (Carlsson, Zomorodian): even more complex because of double multi-filtration.

More general linear semaphore state spaces

- More general semaphores (intersection with the boundary $\partial I^n \subset I^n$ allowed)
- n dining philosophers: Trace space has $2^n - 2$ contractible components!
- Different end points: $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ and iterative calculations
- End complexes rather than end points (allowing processes not to respond..., Herlihy & Cie)

State space components

New light on definition and determination of components of model space X.

Extensions

2a. Semaphores corresponding to non-linear programs:

Path spaces in product of digraphs

Products of digraphs instead of $\vec{I}^n$:
$\Gamma = \prod_{j=1}^{n} \Gamma_j$, state space $X = \Gamma \setminus F$,
F a product of generalized hyperrectangles R^i.

- $\vec{P}(\Gamma)(\mathbf{x}, \mathbf{y}) = \prod \vec{P}(\Gamma_j)(x_j, y_j)$ – homotopy discrete!

Pullback to linear situation

Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$ by (regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j, y_j)$ – an interleaving.
The map $c : \vec{I}^n \to \Gamma, c(t_1, \dots, t_n) = (c_1(t_1), \dots, c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0}, \mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

Homotopy types of interleaving components

Pull back F via c:
$\bar{X} = \vec{I}^n \setminus \bar{F}$, $\bar{F} = \bigcup \bar{R}^i$, $\bar{R}^i = c^{-1}(R^i)$ – honest hyperrectangles!
$i_X : \vec{P}(X) \hookrightarrow \vec{P}(\Gamma)$.
Given a component $C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.
The d-map $c : \bar{X} \to X$ induces a homeomorphism
$c \circ : \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \to i_X^{-1}(C) \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

- C "lifts to X" $\Leftrightarrow \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \neq \emptyset$; if so:
- Analyse $i_X^{-1}(C)$ via $\vec{P}(\bar{X})(\mathbf{0}, \mathbf{1})$.
- Exploit relations between various components.

Special case: $\Gamma = (S^1)^n$ – a torus

State space: A torus with rectangular holes in F:
Investigated by Fajstrup, Goubault, Mimram etal.:
Analyse by language on the alphabet $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ of alive matrices for a delooping of $\Gamma \setminus F$.

HDA: Directed pre-cubical complex

Higher Dimensional Automaton: Pre-cubical complex – like simplicial complex but with cubes as building blocks – with preferred diretions.
Geometric realization X with d-space structure.

Branch points and branch cubes

These complexes have branch points and branch cells – more than one maximal cell with same lower corner vertex.
At branch points, one can cut up a cubical complex into simpler pieces.
Trouble: Simpler pieces may have higher order branch points.

Non-branching complexes

Start with complex without directed loops: After finally many iterations: Subcomplex Y without branch points.

Theorem

$\vec{P}(Y)(\mathbf{x}_0, \mathbf{x}_1)$ is empty or contractible.

Proof.

Such a subcomplex has a preferred diagonal flow and a contraction from path space to the flow line from start to end. □

Branch category

Results in a (complicated) finite branch category $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ on subsets of set of (iterated) branch cells.

Theorem

$\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of that category.

Delooping HDAs

A pre-cubical complex comes with an L_1-length 1-form ω reducing to $\omega = dx_1 + \cdots + dx_n$ on every n-cube.
Integration: L_1-length on rectifiable paths, homotopy invariant.
Defines $l : P(X)(x_0, x_1) \to \mathbf{R}$ and $l_\sharp : \pi_1(X) \to \mathbf{R}$ with kernel K.
The (usual) covering $\tilde{X} \downarrow X$ with $\pi_1(\tilde{X}) = K$ is a directed pre-cubical complex without d- loops.

Theorem (Decomposition theorem)

For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n \in \mathbf{Z}} \vec{P}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$[a].

[a] in the fibres over $\mathbf{x}_0, \mathbf{x}_1$

To conclude

- From a (rather compact) state space model to a finite dimensional trace space model.
- Calculations of invariants (Betti numbers) of path space possible for state spaces of a moderate size.
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors) – linearly.
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  Connections to the work of Herlihy and Rajsbaum: protocol complex etc
- Challenge: Morphisms between HDA $\rightsquigarrow$ d-maps between pre-cubical state spaces $\rightsquigarrow$ functorial maps between trace spaces. Properties? Equivalences?

Want to know more?
Thank you!

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Thank you for your attention!