Spaces of executions as simplicial complexes

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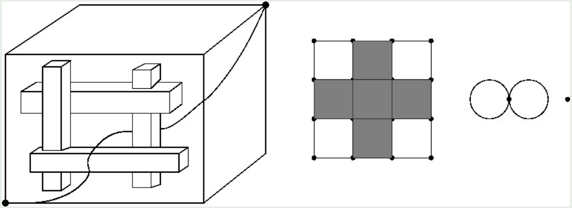
Intro: State space, directed paths and trace space

Problem: How are they related?

Example 1: State space and trace space for a semaphore HDA



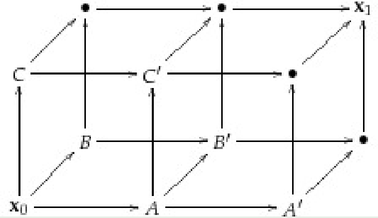
State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

Path space model contained in torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

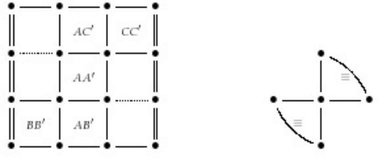
Analogy in standard algebraic topology

Relation between space X and loop space ΩX.

Example 2: State space and trace space for a non-looping cubical complex

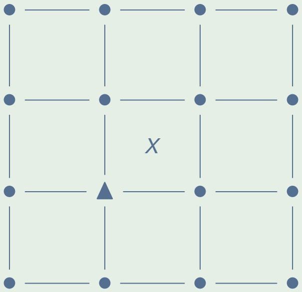


State space: Boundaries of two cubes glued together at common square $AB'C'\bullet$
Branch points at $\mathbf{x}_0$ and A.



Path space model: Prodsimplicial complex contained in $(\partial\Delta^2)^2 \cup \Delta^2$– homotopy equivalent to $S^1 \vee S^1$

Example 3: Torus with a hole



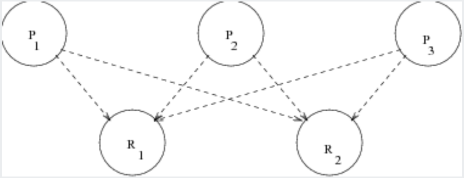
State space: torus with hole X and branch point ▲:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$.

Question: Path space for a torus with hole in higher dimensions?
Forthcoming work, K. Ziemiański.

Mutual exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores

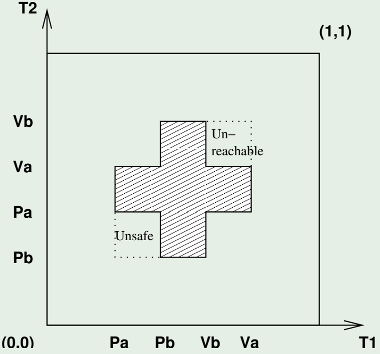
Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

P: probeer; V: verhoog

A geometric model: Schedules in "progress graphs"

Semaphores: The Swiss flag example



PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:

- Hole = isothetic hyperrectangle $R^i =]a^i_1, b^i_1[\times \cdots \times]a^i_n, b^i_n[\subset I^n, 1 \leq i \leq l$: with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
- State space $X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^l R^i$ X inherits a partial order from $\vec{I}^n$. d-paths are order preserving.

More general concurrent programs $\rightsquigarrow$ HDA

Higher Dimensional Automata (HDA, V. Pratt; 1990):

- Cubical complexes: like simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.
- d-paths are order preserving.

Definition

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a, b) = \{p : \vec{I} \to X | \ p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a, b) = \vec{P}(X)(a, b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a, b) \simeq \vec{T}(X)(a, b)$.
- A dihomotopy in $\vec{P}(X)(a, b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a, b), \ t \in I$; ie a path in $\vec{P}(X)(a, b)$.

Aim:

Description of the homotopy type of $\vec{P}(X)(a, b)$ as explicit finite dimensional (prod-)simplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

Tool: Subspaces of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

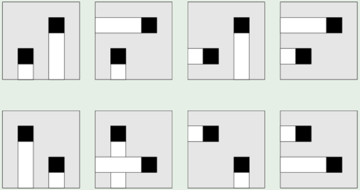
$X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^{l} R^i$; $R^i = [\mathbf{a}^i, \mathbf{b}^i]$; $\mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

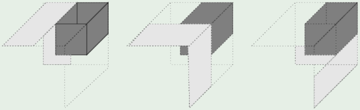
1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – direction j restricted at hole i
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ – Which directions are restricted at which hole?

Examples: two holes in 2D – one hole in 3D (dark)

$M =$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



$M =$ [100] [010] [001]



Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^{R,*}$ no row vector is the zero vector – every hole obstructed in at least one direction

A cover by contractible subspaces

Theorem

1. $$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,*}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible.* *Which is which?*

Proof.

Subspaces X_M, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

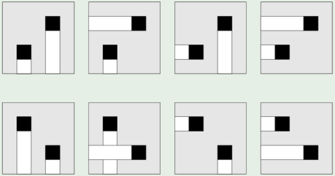
A combinatorial model and its geometric realization
First examples

Combinatorics
poset category
$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^{R,*} \subseteq M_{l,n}$
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ "alive"

Topology:
prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq$
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ – one simplex Δ_{m_i} for every hole

$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$

Examples of path spaces



$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$

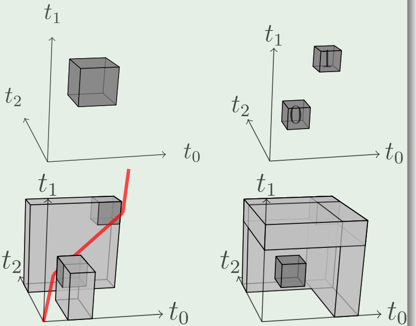
- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

$\supset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

Further examples

State spaces, “alive” matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^{R,*}_{1,n} \setminus \{[1,\dots,1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2. $X = \vec{I}^n \setminus (\vec{J}^n_0 \cup \vec{J}^n_1)$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^{R,*}_{2,n} \setminus$ matrices with a $[1,\dots,1]$-row.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq S^{n-2} \times S^{n-2}$.



$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ alive

$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ dead

Homotopy equivalence between trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ and the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem (A variant of the nerve lemma)

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- $\mathrm{colim}\ \mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, $\mathrm{colim}\ \mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  $\mathrm{hocolim}\ \mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  $\mathrm{hocolim}\ \mathcal{D} \cong \mathrm{hocolim}\ \mathcal{T}^* \cong \mathrm{hocolim}\ \mathcal{T} \cong \mathrm{hocolim}\ \mathcal{E}$.
- Projection lemma:
  $\mathrm{hocolim}\ \mathcal{D} \simeq \mathrm{colim}\ \mathcal{D}$, $\mathrm{hocolim}\ \mathcal{E} \simeq \mathrm{colim}\ \mathcal{E}$.

□

Why prodsimplicial?
rather than simplicial

Good reasons: size!

- We distinguish, for every obstruction, sets $J_i \subset [1:n]$ of restrictions. A joint restriction is of product type $J_1 \times \cdots \times J_l \subset [1:n]^l$, and
  not an arbitrary subset of $[1:n]^l$.
- Simplicial model: a subcomplex of $\Delta^{n^l} - 2^{(n^l)}$ subsimplices.
- Prodsimplicial model: a subcomplex of $(\Delta^n)^l$ of product type $- 2^{(nl)}$ subsimplices.

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL:
  Progress at CEA/LIX-lab.: Goubault, Haucourt, Mimram
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek's CrHom etc) to calculate the homology groups of these chain complexes even for very big complexes: M. Juda (Krakow).
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
  For path-components alone, there are fast "discrete" methods, that also yield representatives in each path component (ALCOOL).

Detection of dead and alive subcomplexes

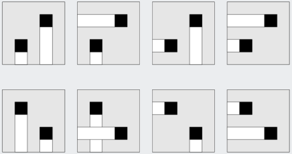
An algorithm starts with deadlocks and unsafe regions!

Allow less = forbid more!

Remove **extended** hyperrectangles

$R^i_j := [0, b^i_1[\times \cdots \times [0, b^i_{j-1}[\times]a^i_j, b^i_j[\times [0, b^i_{j+1}[\times \cdots \times [0, b^i_n[\supset R^i$.

$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j$.



Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.
2. *There is a "**dead**" matrix $N \leq M$, $N \in M^{C,u}_{l,n}$ – every column a unit vector – such that $\bigcap_{n_{ij}=1} R^i_j \neq \emptyset$ – giving rise to a **deadlock** unavoidable from $\mathbf{0}$, i.e., $T(X_N)(\mathbf{0}, \mathbf{1}) = \emptyset$.*

Dead matrices in $D(X)(\mathbf{0}, \mathbf{1})$

Inequalities decide

Decisions: Inequalities – Overlap of intervals

Deadlock algorithm (Fajstrup, Goubault, Raussen) $\rightsquigarrow$:

Theorem

- *$N \in M_{l,n}^{C,u}$ dead $\Leftrightarrow$*
  For all $1 \leq j \leq n$, for all $1 \leq k \leq n$ such that $\exists j' : n_{kj'} = 1$:

$$n_{ij} = 1 \Rightarrow a_j^i < b_j^k.$$

- *$M \in M_{l,n}^{R,*}$ dead $\Leftrightarrow \exists N \in M_{l,n}^{C,u}$ dead, $N \leq M$.*

Definition

$D(X)(\mathbf{0}, \mathbf{1}) := \{P \in M_{l,n} | \exists N \in M_{l,n}^{C,u}, N$ *dead* $: N \leq P\}$.

Maximal alive $\leftrightarrow$ minimal dead

Still alive – not yet dead

- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1}) \subset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ maximal alive matrices.
- Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.
- Connection: $M \in \mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$, $M \leq N$ a succesor (a single 0 replaced by a 1) $\Rightarrow N \in D(X)(\mathbf{0},\mathbf{1})$.

Example: A cube removed from a cube

- $X = \vec{I}^n \setminus \vec{J}^n$, $D(X)(\mathbf{0},\mathbf{1}) = \{[1,\ldots,1]\}$;
- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$: vectors with a single 0;
- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{l,n} \setminus \{[1,\ldots,1]\}$;
- $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1}$.

Open problem: Huge complexes – complexity

Possible antidotes

- l obstructions, n processors:
  $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ is a subcomplex of $(\partial\Delta^{n-1})^l$:
  potentially a huge high-dimensional complex.
- Smaller models? Make use of partial order among the obstructions R^i, and in particular the inherited partial order among their extensions R^i_j with respect to $\subseteq$.
- Consider only saturated matrices in the sense:
  $R^{i_1}_j \subset R^{i_2}_j, m_{i_2 j} = 1 \Rightarrow m_{i_1 j} = 1$.
- Work in progress: yields simplicial complex of far smaller dimension!

Open problem: Variation of end points

Conncection to MD persistence?

Components?!

- So far: $\vec{T}(X)(\mathbf{0},\mathbf{1})$ - fixed end points.
- Now: Variation of $\vec{T}(X)(\mathbf{a},\mathbf{b})$ of start and end point, giving rise to filtrations.
- At which thresholds do homotopy types change?
- How to cut up $X \times X$ into components so that the homotopy type of trace spaces with end point pair in a component is invariant?
- Birth and death of homology classes?
- Compare with multidimensional persistence (Carlsson, Zomorodian).

HDA: Directed cubical complex

Higher Dimensional Automaton: Cubical complex – like simplicial complex but with cubes as building blocks – with preferred diretions.
Geometric realization X with d-space structure.

Branch points and branch cubes

These complexes have branch points and branch cells – more than one maximal cell with same lower corner vertex.
At branch points, one can cut up a cubical complex into simpler pieces.
Trouble: Simpler pieces may have higher order branch points.

Non-branching complexes

Start with complex without directed loops: After finally many iterations: Subcomplex Y without branch points – NB.

Theorem

Y an NB-complex $\Rightarrow \vec{P}(Y)(\mathbf{x}_0, \mathbf{x}_1)$ is empty or contractible.

Proof.

Such a subcomplex has a preferred diagonal flow *leadsto* Contraction from path space to flow line from start to end. □

Branch category

Results in a (complicated) finite branch category $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ on subsets of set of (iterated) branch cells.

Theorem

$\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of that category.

Delooping HDAs

A cubical complex comes with an L_1-length 1-form ω reducing to "diagonal form " $\omega = dx_1 + \cdots + dx_n$ on every n-cube.
Integration: L_1-length on rectifiable paths, homotopy invariant.
Defines $I : P(X)(x_0, x_1) \to \mathbf{R}$ and $I_\sharp : \pi_1(X) \to \mathbf{R}$ with kernel K.
The (usual) covering $\tilde{X} \downarrow X$ with $\pi_1(\tilde{X}) = K$ is a directed cubical complex without d- loops.

Theorem (Decomposition theorem)

For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n \in \mathbf{Z}} \vec{P}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)^a$.

[a] in the fibres over $\mathbf{x}_0, \mathbf{x}_1$

To conclude

Conclusions and challenges

- From a (rather compact) state space model to a finite dimensional trace space model.
- Calculations of invariants (Betti numbers) of path space possible for state spaces of a moderate size.
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors).
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  Connections to the work of Herlihy and Rajsbaum protocol complex etc
- Challenge: Morphisms between HDA $\rightsquigarrow$ d-maps between cubical state spaces $\rightsquigarrow$ functorial maps between trace spaces. Properties? Equivalences?

Want to know more?

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