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Thus

$$\left|\varphi - \frac{f_{n+1}}{f_n}\right| = \frac{1}{\sqrt{5}\, f_n(f_n + \overline{\varphi}^n)},$$

and it follows that, for every $\varepsilon > 0$, the inequality $|\varphi - p/q| < 1/(q^2(\sqrt{5} + \varepsilon))$ has only finitely many rational solutions p/q.

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A New Short Proof of Kneser's Conjecture

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In a 1955 paper [**4**], M. Kneser considered the problem of partitioning the n-element subsets of a $(2n + k)$-element set in such a way that the subsets contained in any fixed class are pairwise intersecting. Kneser observed that such a partition is possible with $k + 2$ classes; indeed, let $1, 2, \ldots, 2n + k$ be the elements of the underlying set, and for each i in this set, let K_i denote the collection of all n-subsets whose least element is i. Then $K_1, K_2, \ldots, K_{k+1}$, and $K_{k+2} \cup \cdots \cup K_{n+k+1}$ are the classes in a desired partition. Moreover, Kneser conjectured that $k + 2$ is the least possible number of classes in such a valid partition. This problem remained open for over twenty years until L. Lovász [**5**] showed, using methods from algebraic topology, that Kneser's conjecture was true. Within weeks of learning of Lovász's proof, I. Bárány [**1**] produced a very short proof of the conjecture by combining the celebrated result of Lusternik, Schnirelman, and Borsuk (LSB) [**2**], [**6**] on sphere covers with D. Gale's theorem [**3**] concerning the even distribution of points on the sphere. The purpose of this note is to provide a short proof of Kneser's conjecture that does not rely on Gale's result.

Let $S^m = \{x \in \mathbb{R}^{m+1} \mid \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^{m+1}$. For any point a in S^m and subset F of S^m, the *distance* from a to F is $\inf_{x \in F} d(a, x)$, where d denotes the Euclidean metric in $\mathbb{R}^{m+1}$. Let $H(a) = \{x \in S^m \mid a \cdot x > 0\}$, the open hemisphere centered at a, and $S(a) = \{x \in S^m \mid a \cdot x = 0\}$, the boundary of $H(a)$, a great $(m-1)$-sphere on S^m. Also, for $\lambda > 0$, let $B(a, \lambda) = \{x \in S^m \mid d(a, x) < \lambda\}$, the open ball of radius λ centered at a in S^m.

The LSB-theorem states that, for any covering of S^m with $m+1$ or fewer closed sets, one of the sets must contain a pair of antipodes. In order to prove Kneser's conjecture, we will require the following slight generalization of this fact.

Lemma. *If S^m is covered with $m+1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodes.*

Proof. We induct on the number t of closed sets in the cover of S^m. The base case $t = 0$ corresponds to a cover of S^m by open sets $U_1, \ldots, U_{m+1}$. Select a *Lebesgue number* for this cover, that is, a positive number λ such that for all x in S^m, the closed ball $\bar{B}(x, \lambda)$ is contained in some U_j. By compactness, there exists a finite collection of points $\{x_i\}$ such that the open balls $B(x_i, \lambda)$ cover S^m. For each j, let F_j denote the union of those $\bar{B}(x_i, \lambda)$ contained in U_j. Then F_j is closed, F_j is a subset of U_j for each j, and together the F_j cover S^m. Therefore, the LSB-theorem implies that one of the F_j, and hence one of the U_j, contains a pair of antipodes.

Thus we may assume that $0 < t < m+1$ and the theorem holds for fewer than t closed sets. We now show it holds for t closed sets. Let $\mathcal{C}$ be a cover of S^m with $m+1$ sets, of which exactly t are closed and the remaining sets are open. Fix a closed set F in $\mathcal{C}$, and suppose that F does not contain a pair of antipodes. Hence its diameter is $2 - \epsilon$ for some $\epsilon > 0$. Let U denote the open set consisting of all points in S^m whose distance from F is less than $\epsilon/2$. Then $(\mathcal{C} \setminus \{F\}) \cup \{U\}$ is a cover of S^m with $m+1$ sets, of which exactly $t-1$ are closed and the remaining sets are open, so by the induction hypothesis some set in this cover contains a pair of antipodes. But by construction U does not contain such a pair, and hence some set in the original cover $\mathcal{C}$ must contain a pair of antipodes, as desired. This completes the inductive step. ■

We are now in position to prove Kneser's conjecture.

Theorem. *If the n-element subsets of a $(2n+k)$-element set are partitioned into $k+1$ classes, then one of the classes must contain a pair of disjoint subsets.*

Proof. Distribute $2n+k$ points on S^{k+1} in general position; thus no $k+2$ points lie on a great k-sphere. Now partition the n-element subsets of these points into $k+1$ classes $A_1, \ldots, A_{k+1}$. For $i = 1, \ldots, k+1$, let U_i denote the set of all points a of S^{k+1} for which $H(a)$ contains an n-subset in the class A_i. It is easy to see that the sets U_i are open, hence $F = S^{k+1} \setminus (U_1 \cup \cdots \cup U_{k+1})$ is closed. Together, F and the U_i are $k+2$ sets covering S^{k+1}, so by the lemma one of the sets contains a pair of antipodes $\pm a$. Can F contain such a pair? No, for if it did $H(a)$ and $H(-a)$ would each contain fewer than n points from our underlying $(2n+k)$-element set, which would mean that at least $k+2$ points lie on the great k-sphere $S(a)$, contradicting the distribution of these points. Therefore, both a and $-a$ lie in U_i for some i. It follows that both $H(a)$ and $H(-a)$ contain n-element subsets in the class A_i, and these subsets are plainly disjoint. ■

In Bárány's proof of Kneser's conjecture, Gale's theorem was used to distribute the $2n+k$ points on S^k in such a way that each open hemisphere contained at least n points. This distribution guaranteed that the sets U_i themselves covered S^k, so that the open version of the LSB-theorem on S^k could be applied. By contrast, our proof distributes the points on S^{k+1} in less restrictive fashion, avoiding the use of Gale's theorem, and then appeals to our modification of the LSB-theorem.

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