# Spaces of directed paths as simplicial complexes

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Homotopy type of path space described by a matrix poset category and realized by a prodsimplicial complex

Algorithmics: Detecting dead and alive subcomplexes/matrices

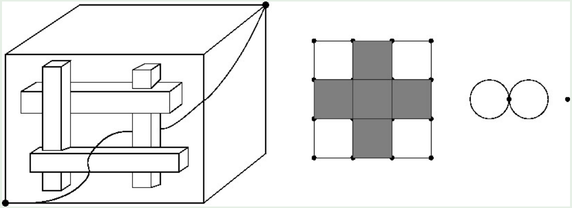
Outlook: How to handle general HDA – with directed loops

Case: Directed loops on a punctured torus (joint with K. Ziemiański)

# Intro: State space, directed paths and trace space

Problem: How are they related?

## Example 1: State space and trace space for a semaphore HDA



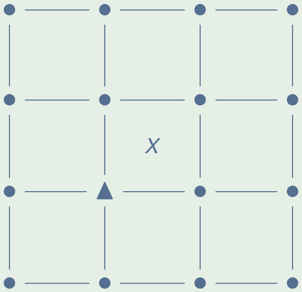
State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

Path space model contained in torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

## Analogy in standard algebraic topology

Relation between space $X$ and loop space $\Omega X$.

## Example 2: Punctured torus




State space: Punctured torus $X$ and branch point ▲:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$.

Question: Path space for a punctured torus in higher dimensions?
Joint work with K. Ziemiański.

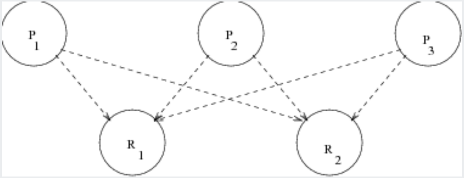
# Why bother? Concurrency
## Definition from Wikipedia

### Concurrency

- In computer science, concurrency is a property of systems in which several computations are executing simultaneously, and potentially interacting with each other.
- The computations may be executing on multiple cores in the same chip, preemptively time-shared threads on the same processor, or executed on physically separated processors.
- A number of mathematical models have been developed for general concurrent computation including Petri nets, process calculi, the Parallel Random Access Machine model, the Actor model and the Reo Coordination Language.
- Specific applications to static program analysis – design of automated tools to test correctness etc. of a concurrent program regardless of specific timed execution.

## Mutual exclusion

occurs, when $n$ processes $P_i$ compete for $m$ resources $R_j$.




Only $k$ processes can be served at any given time.

## Semaphores

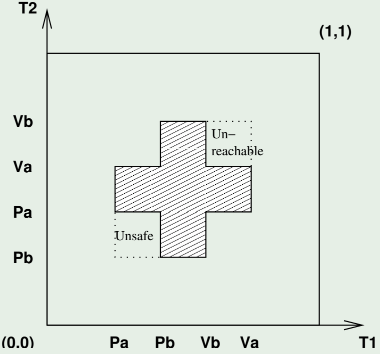
Semantics: A processor has to lock a resource and to relinquish the lock later on!

**Description/abstraction:** $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

$P$: prolaag; $V$: verhogen

# A geometric model: Schedules in "progress graphs"

## Semaphores: The Swiss flag example




PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

### The state space

A linear PV-program is modeled as the complement of a forbidden region $F$ consisting of a number of holes in an $n$-cube:

- Hole = isothetic hyperrectangle
  $R^i = ]a_1^i, b_1^i[ \times \cdots \times ]a_n^i, b_n^i[ \subset I^n, 1 \leq i \leq l$:
  with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
- State space $X = \vec{I}^n \setminus F, \ F = \bigcup_{i=1}^{l} R^i$
  $X$ inherits a partial order from $\vec{I}^n$.
  d-paths are order preserving.

### More general concurrent programs ⇝ HDA

Higher Dimensional Automata (HDA, V. Pratt; 1990):

- Cubical complexes: like simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks[a]
- d-paths are order preserving.

[a] We tacitly suppress labels

# Spaces of d-paths/traces – up to dihomotopy

## A general framework. Aims.

### Definition

- $X$ a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in $X$ (continuous and "order-preserving") from $a$ to $b$.
- $\vec{P}(X)(a, b) = \{p : \vec{I} \to X | \ p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a, b) = \vec{P}(X)(a, b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a, b) \simeq \vec{T}(X)(a, b)$.
- A dihomotopy in $\vec{P}(X)(a, b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a, b), \ t \in I$; ie a path in $\vec{P}(X)(a, b)$.

### Aim:

Description of the homotopy type of $\vec{P}(X)(a, b)$ as explicit finite dimensional (prod-)simplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

# Tool: Subspaces of $X$ and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

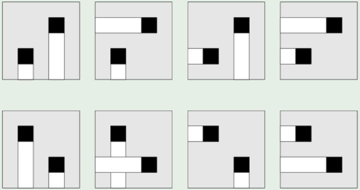
$X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^{l} R^i$; $R^i = ]\mathbf{a}^i, \mathbf{b}^i[$; $\mathbf{0}, \mathbf{1}$ the two corners in $I^n$.

## Definition

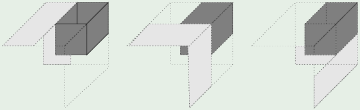
1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – *direction $j$ restricted at hole $i$*
2. $M$ a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ – *Which directions are restricted at which hole?*

## Examples: Two holes in 2D – One hole in 3D (dark)

$M =$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



$M =$ $[100]$ $[010]$ $[001]$



# Covers by contractible (or empty) subspaces

## Bookkeeping with binary matrices

### Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^{R,*}$ no row vector is the zero vector – *every hole obstructed in at least one direction*

### A cover by contractible subspaces

#### Theorem

1. $$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,*}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible.* *Which is which?*

#### Proof.

Subspaces $X_M$, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

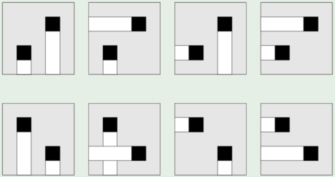
**Combinatorics:**
**poset category**

$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^{R,*} \subseteq M_{l,n}$
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ “alive”

**Topology:**
**prodsimplicial complex**
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq$
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ – one simplex $\Delta_{m_i}$
for every hole

$$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$$

## Examples of path spaces



$\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$

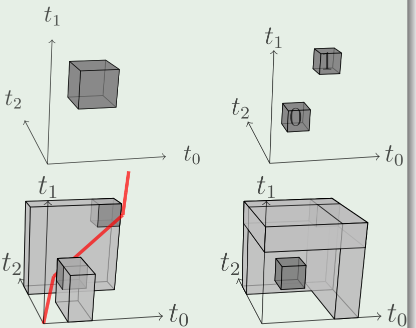
- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

$\supset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

# Further examples

## State spaces, “alive” matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{1,n}^{R,*} \setminus \{[1,\dots,1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2. $X = \vec{I}^n \setminus (\vec{J}_0^n \cup \vec{J}_1^n)$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{2,n}^{R,*} \setminus$ matrices with a $[1,\dots,1]$-row.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq S^{n-2} \times S^{n-2}$.




$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

alive dead

# Why prodsimplicial?
## rather than simplicial

- We distinguish, for every obstruction, sets $J_i \subset [1:n]$ of restrictions. A joint restriction is of product type $J_1 \times \cdots \times J_l \subset [1:n]^l$, and not an arbitrary subset of $[1:n]^l$.
- Simplicial model: a subcomplex of $\Delta^{n^l}$ – $2^{(n^l)}$ subsimplices.
- Prodsimplicial model: a subcomplex of $(\Delta^n)^l$ – $2^{(nl)}$ subsimplices.

# Homotopy equivalence between path space $\vec{P}(X)(\mathbf{0},\mathbf{1})$ and prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

**Theorem (A variant of the nerve lemma)**

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

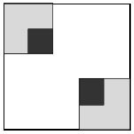
**Proof.**

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  hocolim $\mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  hocolim $\mathcal{D} \simeq$ hocolim $\mathcal{T}^* \simeq$ hocolim $\mathcal{T} \simeq$ hocolim $\mathcal{E}$.
- Projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

□

## Allow less = forbid more!

Remove **extended** hyperrectangles $R^i_j$ $:= [0, b^i_1[\times \cdots \times [0, b^i_{j-1}[\times ]a^i_j, 1] \times [0, b^i_{j+1}[\times \cdots \times [0, b^i_n[ \supset R^i$.



$$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j.$$

## Theorem

*The following are equivalent:*

1. $\vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.
2. *There is a "**dead**" matrix* $N \leq M, N \in M^{C,u}_{l,n}$ *such that* $\bigcap_{n_{ij}=1} R^i_j \neq \emptyset$ *– giving rise to a **deadlock** unavoidable from* $\mathbf{0}$, *i.e.,* $T(X_N)(\mathbf{0}, \mathbf{1}) = \emptyset$.
   $M^{C,u}_{l,n}$: *every column a unit vector – every direction obstructed once.*

# Dead matrices in $D(X)(\mathbf{0},\mathbf{1})$

## Inequalities decide

**Decisions: Inequalities**

Deadlock algorithm (Fajstrup, Goubault, Raussen) $\rightsquigarrow$:

**Theorem**

- $N \in M_{l,n}^{C,u}$ *dead* $\Leftrightarrow$
  *For all* $1 \leq j \leq n$, *for all* $1 \leq k \leq n$ *such that* $\exists j' : n_{kj'} = 1$:

$$n_{ij} = 1 \Rightarrow a_j^i < b_j^k.$$

- $M \in M_{l,n}^{R,*}$ *dead* $\Leftrightarrow \exists N \in M_{l,n}^{C,u}$ *dead*, $N \leq M$.

**Definition**

$D(X)(\mathbf{0},\mathbf{1}) := \{P \in M_{l,n} | \exists N \in M_{l,n}^{C,u}, N \text{ dead} : N \leq P\}$.

**A cube with a cubical hole**

- $X = \vec{I}^n \setminus \vec{J}^n$
- $D(X)(\mathbf{0},\mathbf{1}) = \{[1,\ldots,1]\} = M_{1,n}^{C,u}$.

# Maximal alive $\leftrightarrow$ minimal dead

## Still alive – not yet dead

- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1}) \subset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ maximal alive matrices.
- Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.
- Connection: $M \in \mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$, $M \leq N$ a succesor (a single 0 replaced by a 1) $\Rightarrow N \in D(X)(\mathbf{0},\mathbf{1})$.

## A cube with a cubical hole

- $X = \vec{I}^n \setminus \vec{J}^n$, $D(X)(\mathbf{0},\mathbf{1}) = \{[1,\ldots,1]\}$;
- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$: vectors with a single 0;
- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{l,n} \setminus \{[1,\ldots,1]\}$;
- $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1}$.

# From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

## Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

### Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

### Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL at CEA/LIX-lab.: Goubault, Haucourt, Mimram
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek's CrHom etc) to calculate the homology groups of these chain complexes even for quite big complexes: M. Juda (Krakow).
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
  For path-components alone, there are fast "discrete" methods, that also yield representatives in each path component (ALCOOL).

# Open problem: Huge complexes – complexity

## Huge prodsimplicial complexes

$l$ obstructions, $n$ processors:
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ is a subcomplex of $(\partial\Delta^{n-1})^l$:
potentially a huge high-dimensional complex.

## Possible antidotes

- Smaller models? Make use of partial order among the obstructions $R^i$, and in particular the inherited partial order among their extensions $R^i_j$ with respect to $\subseteq$.
- Work in progress: yields often simplicial complex of far smaller dimension!

## Components?!

- So far: $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ - fixed end points.
- Now: Variation of $\vec{T}(X)(\mathbf{a}, \mathbf{b})$ of start and end point, giving rise to filtrations.
- At which thresholds do homotopy types change?
- How to cut up $X \times X$ into components so that the homotopy type of trace spaces with end point pair in a component is invariant?
- Birth and death of homology classes?
- Compare with multidimensional persistence (Carlsson, Zomorodian).

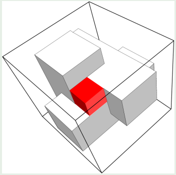
# Extensions

## 1. Obstruction hyperrectangles intersecting the boundary of $I^n$ – why?

### More general linear semaphore state spaces

- More general semaphores (intersection with the boundary $\partial I^n \subset I^n$ allowed)
- $n$ dining philosophers: Trace space has $2^n - 2$ contractible components!
- Different end points: $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ and iterative calculations
- End complexes rather than end points (allowing processes not to respond..., Herlihy & Cie)

### Dining philosophers

## Path spaces in product of digraphs

Products of digraphs instead of $\vec{I}^n$:
$\Gamma = \prod_{j=1}^n \Gamma_j$, state space $X = \Gamma \setminus F$,
$F$ a product of generalized hyperrectangles $R^i$.

- $\vec{P}(\Gamma)(\mathbf{x}, \mathbf{y}) = \prod \vec{P}(\Gamma_j)(x_j, y_j)$ – homotopy discrete!

## Pullback to linear situation

Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$ by (regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j, y_j)$ – an interleaving.
The map $c : \vec{I}^n \to \Gamma, c(t_1, \dots, t_n) = (c_1(t_1), \dots, c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0}, \mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

### Homotopy types of interleaving components

Pull back $F$ via $c$:
$\bar{X} = \vec{I}^n \setminus \bar{F}, \bar{F} = \bigcup \bar{R}^i, \bar{R}^i = c^{-1}(R^i)$ – honest hyperrectangles!
$i_X : \vec{P}(X) \hookrightarrow \vec{P}(\Gamma)$.
Given a component $C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.
The d-map $c : \bar{X} \to X$ induces a homeomorphism
$c\circ : \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \to i_X^{-1}(C) \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

- $C$ "lifts to $X$" $\Leftrightarrow \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \neq \emptyset$; if so:
- Analyse $i_X^{-1}(C)$ via $\vec{P}(\bar{X})(\mathbf{0}, \mathbf{1})$.
- Exploit relations between various components.

### Special case: $\Gamma = (S^1)^n$ – a torus

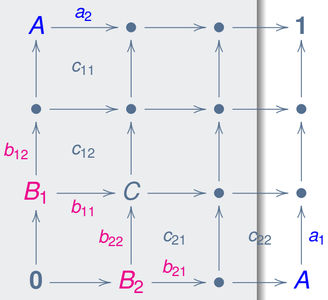
State space: A torus with rectangular holes in $F$:
Investigated by Fajstrup, Goubault, Mimram etal.:
Analyse by language on the alphabet $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ of alive matrices for a one-fold delooping of $\Gamma \setminus F$.

## HDA: Directed pre-cubical complex

Higher Dimensional Automaton: Pre-cubical complex – like simplicial complex but with cubes as building blocks – with preferred diretions.
Geometric realization $X$ with d-space structure.

## Branch points and branch cubes

These complexes have branch points and branch cells – more than one maximal cell with same lower corner vertex.
At branch points, one can cut up a cubical complex into simpler pieces.
Trouble: Simpler pieces may have higher order branch points.



### Non-branching complexes

Start with complex without directed loops: After finally many iterations: Subcomplex $Y$ without branch points.

### Theorem

*$\vec{P}(Y)(\mathbf{x}_0, \mathbf{x}_1)$ is empty or contractible.*

### Proof.

Such a subcomplex has a preferred diagonal flow and a contraction from path space to the flow line from start to end. □

### Branch category

Results in a (complicated) finite branch category $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ on subsets of set of (iterated) branch cells.

### Theorem

*$\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of that category.*

### Delooping HDAs

A pre-cubical complex comes with an $L_1$-length 1-form $\omega$ reducing to $\omega = dx_1 + \cdots + dx_n$ on every $n$-cube.
Integration: $L_1$-length on rectifiable paths, homotopy invariant.
Defines $I : P(X)(x_0, x_1) \to \mathbf{R}$ and $I_\sharp : \pi_1(X) \to \mathbf{R}$ with kernel $K$.
The (usual) covering $\tilde{X} \downarrow X$ with $\pi_1(\tilde{X}) = K$ is a directed pre-cubical complex without d- loops.

### Theorem (Decomposition theorem)

*For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n\in\mathbf{Z}} \vec{P}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$*[a].

[a] in the fibres over $\mathbf{x}_0, \mathbf{x}_1$

# Case: d-paths on a punctured torus

## Punctured torus and $n$-space

$n$-torus $T^n = \mathbf{R}^n/_{\mathbf{Z}^n}$.

forbidden region $F^n = ([\frac{1}{4}, \frac{3}{4}]^n + \mathbf{Z}^n)/_{\mathbf{Z}^n} \subset T^n$.

punctured torus $Q^n = T^n \setminus F^n \simeq T^n_{(n-1)}$

punctured $n$-space $\tilde{Q}^n = \mathbf{R}^n \setminus ([\frac{1}{4}, \frac{3}{4}]^n + \mathbf{Z}^n) \simeq \mathbf{R}^n_{(n-1)}$

with d-paths from quotient map $\mathbf{R}^n \downarrow T^n$.

## Aim: Describe the homotopy type of $\vec{P}(Q) = \vec{P}(Q)(\mathbf{0}, \mathbf{0})$

$\vec{P}(Q) \hookrightarrow \Omega Q(\mathbf{0}, \mathbf{0}) \rightsquigarrow$ disjoint union $\vec{P}(Q) = \bigsqcup_{\mathbf{k} \geq \mathbf{0}} \vec{P}(\mathbf{k})(Q)$
with multiindex = multidegree $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{Z}^n_+, k_i \geq 0$.
$\vec{P}(\mathbf{k})(Q) \cong \vec{P}(\tilde{Q}^n)(\mathbf{0}, \mathbf{k}) =: Z(\mathbf{k})$.

# Path spaces as colimits

## Category $\mathcal{J}(n)$

Poset category of proper non-empty subsets of $[1:n]$ with inclusions as morphisms.
Via characteristic functions isomorphic to the category of non-identical bit sequences of length $n$: $\varepsilon = (\varepsilon_1, \dots \varepsilon_n) \in \mathcal{J}(n)$.
$B\mathcal{J}(n) \cong \partial\Delta^{n-1} \cong S^{n-2}$.

## Definition

$U_\varepsilon(\mathbf{k}) := \{\mathbf{x} \in \mathbf{R}^n | \ \varepsilon_j = 1 \Rightarrow x_j \leq k_j - 1 \text{ or } \exists i : x_i \geq k_i\}$
$Z_\varepsilon(\mathbf{k}) := \vec{P}(U_\varepsilon(\mathbf{k}))(\mathbf{0}, \mathbf{k})$.

## Lemma

$Z_\varepsilon(\mathbf{k}) \simeq Z(\mathbf{k} - \varepsilon)$.

## Theorem

$Z(\mathbf{k}) = \mathrm{colim}_{\varepsilon \in \mathcal{J}(n)} Z_\varepsilon(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z_\varepsilon(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon)$.

# An equivalent homotopy colimit construction

## Inductive homotopy colimites

Using the category $\mathcal{J}(n)$ construct for $\mathbf{k} \in \mathbf{Z}^n, \mathbf{k} \geq \mathbf{0}$:

- $X(\mathbf{k}) = *$ if $\prod_1^n k_i = 0$;
- $X(\mathbf{k}) = \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} X(\mathbf{k} - \varepsilon)$.

By construction $\mathbf{k} \leq \mathbf{l} \Rightarrow X(\mathbf{k}) \subseteq X(\mathbf{l})$; $X(\mathbf{1}) \cong \partial\Delta^{n-1}$.

## Inductive homotopy equivalences

$q(\mathbf{k}) : Z(\mathbf{k}) \to X(\mathbf{k})$:

- $\prod_1^n k_i = 0 \Rightarrow Z(\mathbf{k})$ contractible, $X(\mathbf{k}) = *$
- $q(\mathbf{k}) = \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} q(\mathbf{k} - \varepsilon) : Z(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon) \to \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} X(\mathbf{k} - \varepsilon) = X(\mathbf{k})$.

# Homology and cohomology of space $Z(\mathbf{k})$ of d-paths

## Definition

- $\mathbf{l} \ll \mathbf{m} \in \mathbf{Z}_+^n \Leftrightarrow l_j < m_j, 1 \leq j \leq n$.
- $\mathcal{O}^n = \{(\mathbf{l}, \mathbf{m}) | \, \mathbf{l} \ll \mathbf{m}$ or $\mathbf{m} \ll \mathbf{l}\} \subset \mathbf{Z}_+^n \times \mathbf{Z}_+^n$.
- $\mathbf{B}(\mathbf{k}) := \mathbf{Z}_+^n(\leq \mathbf{k}) \times \mathbf{Z}_+^n(\leq \mathbf{k}) \setminus \mathcal{O}^n$ – unordered pairs
- $\mathcal{I}(\mathbf{k}) :=< \mathbf{lm} | \, (\mathbf{l}, \mathbf{m}) \in \mathbf{B}(\mathbf{k}) > \leq \mathbf{Z} < \mathbf{Z}_+^n(\leq \mathbf{k}) >$.

## Theorem

*For $n > 2$, $H^*(Z(\mathbf{k})) = \mathbf{Z} < \mathbf{Z}_+^n(\leq \mathbf{k}) > /_{\mathcal{I}(\mathbf{k})}$.*
*All generators have degree $n-2$.*
*$H_*(Z(\mathbf{k})) \cong H^*(Z(\mathbf{k}))$ as abelian groups.*

## Proof

Spectral sequence argument, using projectivity of the functor $H_* : \mathcal{J}(n) \to \mathbf{Ab}_*, \; \mathbf{k} \mapsto H_*(Z(\mathbf{k}))$.

### Cube sequences

$[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^r = \mathbf{l}] \in A^n_{r(n-2)}(\mathbf{l})$
of size $\mathbf{l} \in \mathbf{Z}^n_+$, length $r$ and degree $r(n-2)$.
$A^n_*(*)$ the free abelian group generated by all cube sequences.
$A^n_*(\leq \mathbf{k}) := \bigoplus_{\mathbf{l} \leq \mathbf{k}} A^n_*(\mathbf{l})$.
$H_{r(n-2)}(Z(\mathbf{k})) \cong A^n_{r(n-2)}(\leq \mathbf{k})$
generated by cube sequences of length $r$ and size $\leq \mathbf{k}$.

### Betti numbers of $Z(\mathbf{k})$

#### Theorem

$n = 2$: $\beta_0 = \binom{k_1+k_2}{k_1}$; $\beta_j = 0,\ j > 0$;
$n > 2$: $\beta_0 = 1,\ \beta_{i(n-2)} = \prod_1^n \binom{k_j}{i},\ \beta_j = 0$ *else.*

#### Corollary

1. *Small homological dimension of* $Z(\mathbf{k})$: $(\min_j k_j)(n-2)$.
2. *For* $\mathbf{k} = (k, \ldots, k),\ \beta_i(Z(\mathbf{k})) = \beta_{k(n-2)-i}(Z(\mathbf{k}))$.

# Generalization. "Explanation"

- The result can be stated and generalized for a complex $T^n_{(n-1)} \subset K \subset T^n$ – with universal cover $\mathbf{R}^n_{(n-1)} \subset \tilde{K} \subset \mathbf{R}^n$. Homology is generated by cube sequences $[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^r = \mathbf{I}]$ such that the cells $[\mathbf{a}^i - \mathbf{1}, \mathbf{a}^i] \not\subset \tilde{K}$.
- A cube sequence $\mathbf{a}^*$ is maximal if it is not properly contained in another cube sequence with same endpoints.
- A maximal cube sequence $\mathbf{a}^*$ gives rise to a subspace $\vec{P}(\mathbf{a}^*)(\mathbf{0},\mathbf{k}) \subset \vec{P}(\tilde{K})(\mathbf{0},\mathbf{k})$ – concatenation of paths on boundary of cubes $[\mathbf{a}^i - \mathbf{1}, \mathbf{a}^i]$ and contractible path spaces.
- $Y(\mathbf{k}) = \bigcup_{\mathbf{a}^*} \vec{P}(\mathbf{a}^*)(\mathbf{0},\mathbf{k}),\ \mathbf{a}^*$ maximal. Then also $Y(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Y(\mathbf{k} - \varepsilon)$ and $Y(\mathbf{k})$ contractible if $\prod_i k_i = 0$.
- Hence $Y(\mathbf{k}) \simeq X(\mathbf{k}) \simeq Z(\mathbf{k})$.
- $\vec{P}(\mathbf{a}^*)(\mathbf{0},\mathbf{k}) \subset \vec{P}(\tilde{K})(\mathbf{0},\mathbf{k})$ induces an injection $H^*(\vec{P}(\mathbf{a}^*)(\mathbf{0},\mathbf{k})) \cong H^*((S^{n-2})^r) \to H^*(\vec{P}(\tilde{K})(\mathbf{0},\mathbf{k}))$.

# To conclude

## Conclusions and challenges

- From a (rather compact) state space model (shape of data) to a finite dimensional trace space model (represent shape).
- Calculations of invariants (Betti numbers) of path space possible for state spaces of a moderate size (measuring shape).
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors); still: curse of dimensionality.
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  Connections to the work of Herlihy and Rajsbaum – protocol complex etc
- Challenge: Morphisms between HDA $\rightsquigarrow$ d-maps between cubical state spaces $\rightsquigarrow$ functorial maps between trace spaces. Properties? Equivalences?

# Want to know more?

## Books

- Kozlov, Combinatorial Algebraic Topology, Springer, 2008.
- Grandis, Directed Algebraic Topology, Cambridge UP, 2009.

## Articles

- MR, Simplicial models for trace spaces, AGT **10** (2010), 1683 – 1714.
- MR, Execution spaces for simple HDA, Appl. Alg. Eng. Comm. Comp. **23** (2012), 59 – 84.
- MR, Simplicial models for trace spaces II: General Higher Dimensional Automata, AGT **12** (2012), 1741 – 1761.
- Fajstrup, Trace spaces of directed tori with rectangular holes, Aalborg University Research Report R-2011-08.
- Fajstrup et al., Trace Spaces: an efficient new technique for State-Space Reduction, Proceedings ESOP, Lect. Notes Comput. Sci. **7211** (2012), 274 – 294.
- Rick Jardine, Path categories and resolutions, Homology, Homotopy Appl. **12** (2010), 231 – 244.

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