Simplicial models for trace spaces

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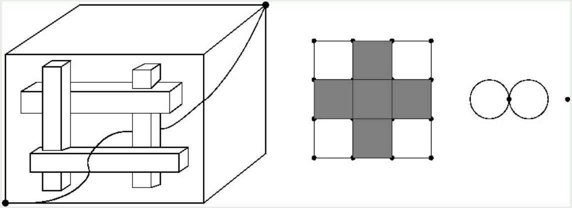
Content

- Higher Dimensional Automata: Examples of state spaces and associated path spaces
- Motivation: Concurrency
- A simple case: State spaces and path spaces related to linear PV-programs
- Tool: Cutting up path spaces into contractible subspaces
- Homotopy type of path space described by a matrix poset category and realized by a prodsimplicial complex
- Algorithmics: Detecting dead and alive subcomplexes/matrices
- Outlook: How to handle general HDA.

Example 1: State space and trace space for a semaphore space



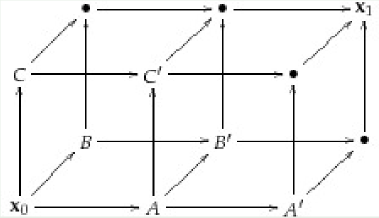
State space = a 3D cube $\vec{I}^3 \setminus F$ minus 4 box obstructions

Path space model contained in a torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

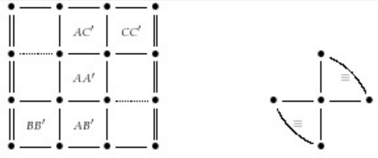
Intro: State space and trace space

Pre-cubical set as state space

Example 2: State space and trace space for a non-looping semi-cubical complex

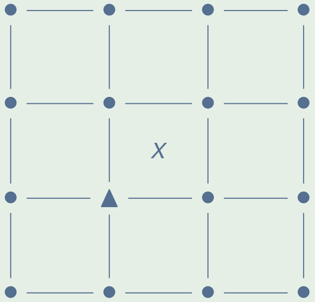


State space: Boundaries of two cubes glued together at common square $AB'C'$ •



Path space model: Prodsimplicial complex contained in torus $(\partial\Delta^2)^2$– homotopy equivalent to $S^1 \vee S^1$

Example 3: Torus with a hole



State space with hole:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

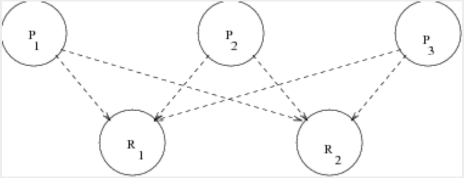
Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$

Motivation: Concurrency

Semaphores: A simple model for mutual exclusion

Mutual exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores

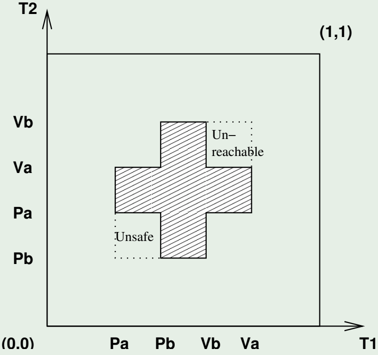
Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

P: pakken; V: vrijlaten

A geometric model: Schedules in "progress graphs"

The Swiss flag example



PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube I^n:

Hole = isothetic hyperrectangle
$R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[, 1 \leq i \leq l$, in an n-cube:
with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
State space $X = \vec{I}^n \setminus F, \; F = \bigcup_{i=1}^{l} R^i$
X inherits a partial order from $\vec{I}^n$.

More general (PV)-programs:

- Replace $\vec{I}^n$ by a product $\Gamma_1 \times \cdots \times \Gamma_n$ of digraphs.
- Holes have then the form $p_1^i((0,1)) \times \cdots \times p_n^i((0,1))$ with $p_j^i : \vec{I} \to \Gamma_j$ a directed injective (d-)path.
- Pre-cubical complexes: like pre-simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.

Spaces of d-paths/traces – up to dihomotopy
the interesting spaces

Definition

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a,b) = \{p : \vec{I} \to X |\ p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
- A dihomotopy on $\vec{P}(X)(a,b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a,b),\ t \in I$; ie a path in $\vec{P}(X)(a,b)$.

Aim:

Description of the homotopy type of $\vec{P}(X)(a,b)$ as explicit finite dimensional prodsimplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

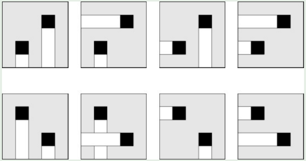
Tool: Covers of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$
by contractible or empty subspaces

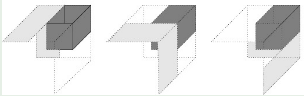
$X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^{l} R^i$; $R^i = [\mathbf{a}^i, \mathbf{b}^i]$; $\mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

$$X_{j_1,\dots,j_l} = \{x \in X \mid \forall i : x_{j_i} \leq a^i_{j_i} \vee \exists k : x_k \geq b^i_k\}$$
$$= \{x \in X \mid \forall i : x \leq \mathbf{b}^i \Rightarrow x_{j_i} \leq a^i_{j_i}\}, \qquad 1 \leq j_i \leq n.$$

Examples:





A cover:

$$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{1 \leq j_1,\dots,j_l \leq n} \vec{P}(X_{j_1,\dots,j_l})(\mathbf{0},\mathbf{1}).$$

More intricate subspaces as intersections

either empty or contractible

Definition

$\emptyset \neq J_1, \dots, J_l \subseteq [1:n]$:

$$\begin{aligned} X_{J_1,\dots,J_l} &= \bigcap_{j_i \in J_i} X_{j_1,\dots,j_l} \\ &= \{x \in X \mid \forall i, j_i \in J_i : x \leq \mathbf{b}^i \Rightarrow x_{j_i} \leq a^i_{j_i}\} \end{aligned}$$

Theorem

Every path space $\vec{P}(X_{J_1,\dots,J_l})(\mathbf{0},\mathbf{1})$ is either empty or contractible.

Proof.

relies on: Subspaces $X_{J_1,\dots,J_l}$ are closed under $\vee$ = l.u.b. □

Question:

For which $J_1, \dots, J_l \subseteq [1:n]$ is $\vec{P}(X_{J_1,\dots,J_l})(\mathbf{0},\mathbf{1}) \neq \emptyset$?

Combinatorics: Bookkeeping with binary matrices

Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices
$M^R_{l,n}$ no row vector is the zero vector
$M^C_{l,n}$ every column vector is a unit vector

Correspondences

$$
\begin{array}{rcl}
\text{Index sets} & \leftrightarrow & \text{Matrix sets} \\
(\mathcal{P}([1:n]))^l & \leftrightarrow & M_{l,n} \\
J = (J_1, \dots, J_l) & \mapsto & M^J = (m_{ij}),\ m_{ij} = 1 \Leftrightarrow j \in J_i \\
J^M & \leftarrow & M \quad J^M_i = \{j \mid m_{ij} = 1\} \\
\text{l-tuples of subsets} \neq \emptyset & \leftrightarrow & M^R_{l,n} \\
\{(K_1, \dots, K_l) \mid [1:n] = \bigsqcup K_i\} & \leftrightarrow & M^C_{l,n}
\end{array}
$$

Question rephrased

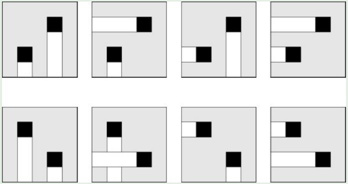
$$X_M := X_{J_M}, \qquad \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \vec{P}(X_{J_M})(\mathbf{0}, \mathbf{1}) \neq \emptyset ?$$

Combinatorics: poset category –
$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M^R_{l,n} \subseteq M_{l,n}$
$J \leftrightarrow M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

Topology: prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta^{|J_1|-1}_{J_1} \times \cdots \times \Delta^{|J_l|-1}_{J_l} \subseteq \mathbf{T}(X)(\mathbf{0},\mathbf{1})$

$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$

First examples



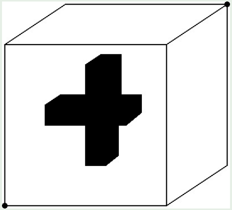
$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$

- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

$\supset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

Further examples

State spaces and "alive" matrices

1. $X = \vec{I}^n \setminus \vec{J}^n$
2. 

1.
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{1,n} \setminus \{[1,\ldots 1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2.
   - $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\}$.
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = \{M \in M^R_{l,n} \mid \exists N \in \mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) : M \leq N\}$
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) =$ 3 diagonal squares $\subset (\partial\Delta^2)^2 = T^2 \simeq S^1$.

Many more examples in Goubault's talk!

Homotopy equivalence between trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ and the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to$ **Top**:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(M) = \Delta_{J_1}^{|J_1|-1} \times \cdots \times \Delta_{J_l}^{|J_l|-1} = \Delta_{J_M}$,
  $\mathcal{T}(M) = *$
- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  hocolim $\mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  hocolim $\mathcal{D} \cong$ hocolim $\mathcal{T}^* \cong$ hocolim $\mathcal{T} \cong$ hocolim $\mathcal{E}$.
- Projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

□

Why prodsimplicial?
rather than simplicial

- We distinguish, for every obstruction, sets J_i of restrictions. A joint restriction is of type $J_1 \times \cdots \times J_l$, and not an arbitrary subset of $[1:n]^l$.
- Prodsimplicial and simplicial model (nerve of category) have the same number of vertices ($\leq n^l$) and dimension ($\leq (n-1)(l-1)-1$).
- The number of cells is of different orders:

prodsimplicial	2^{nl}
simplicial	$2^{(n^l)}$

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL:
  Progress at CEA/LIX-lab.: Goubault etal
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek CrHom etc) to calculate the homology groups of these chain complexes even for very big complexes.
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
  For path-components alone, there are faster "discrete" methods, that also yield representatives in each path component: Goubault etal.
- Even when "exponential explosion" prevents precise calculations, inductive determination (round by round) of general properties ((simple) connectivity) may be possible.

Detection of dead and alive subcomplexes

An algorithm starts with deadlocks and unsafe regions!

Allow less = forbid more!

Remove **extended** hyperrectangles R^i_j
$:= [0, b^i_1[\times \cdots \times [0, b^i_{j-1}[\times]a^i_j, b^i_j[\times [0, b^i_{j+1}[\times \cdots \times [0, b^i_n[\supset R^i$.

$$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j.$$

Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.
2. *There is a map $i : [1 : n] \to [1 : l]$ such that $m_{i(j),j} = 1$*[a] *and such that $\bigcap_{1 \le j \le n} R^{i(j)}_j \neq \emptyset$ – giving rise to a* ***deadlock*** *unavoidable from* **0**.

[a] corresponding to a matrix $M(i) \in M^C_{l,n}$ with $M(i) \le M$

Dead or alive?

Consider $\Psi : M_{l,n} \to \mathbf{Z}/2$, $\Psi(M) = 1 \Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$.

- Ψ is order preserving, in particular: $\Psi^{-1}(0), \Psi^{-1}(1)$ are closed in opposite senses: $M \leq N : \Psi(N) = 0 \Rightarrow \Psi(M) = 0; \Psi(M) = 1 \Rightarrow \Psi(N) = 1$ (thus $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ prodsimplicial).
- $\Psi(M) = 1 \Leftrightarrow \exists N \in M^C_{l,n}$ such that $N \leq M, \Psi(N) = 1$

$$D(X)(\mathbf{0},\mathbf{1}) = \{N \in M^C_{l,n} | \Psi(N) = 1\} \text{ – dead}$$

$$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = \{M \in M^R_{l,n} | \Psi(M) = 0\} \text{ – alive}$$

Maximal alive – minimal dead

Still alive – not yet dead

- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1}) \subset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ maximal alive matrices.
- Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.
- $D_{\min}(X)(\mathbf{0},\mathbf{1}) = D(X)(\mathbf{0},\mathbf{1}) \cap M_{l,n}^C$ minimal dead matrices.
- Connection: $M \in \mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$, $M \leq N$ a succesor (a single 0 replaced by a 1) $\Rightarrow N \in D_{\min}(X)(\mathbf{0},\mathbf{1})$.

A cube removed from a cube

- $X = \vec{I}^n \setminus \vec{J}^n$, $D(X)(\mathbf{0},\mathbf{1}) = \{[1,\ldots,1]\}$;
- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$: vectors with a single 0;
- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{l,n}^R \setminus \{[1,\ldots,1]\}$;
- $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1}$.

Decisions: Inequalities

- Enough to decide among the l^n matrices in $M_{l,n}^C$.
- A matrix $M \in M_{l,n}^C$ is described by a (choice) map

$$i : [1:n] \to [1:l], m_{i(j),j} = 1.$$

- Deadlock algorithm $\rightsquigarrow$ inequalities:

$$M \in D(X)(\mathbf{0},\mathbf{1}) \Leftrightarrow a_j^{i(j)} < b_j^{i(k)} \text{ for all } 1 \leq j,k \leq n.$$

- Algorithmic organisation: Choice maps with the same image give rise to the same upper bounds b_j^*.

From $D(X)$ to $\mathcal{C}_{max}(X)$

Minimal transversals in hypergraphs (simplicial complexes)

Incremental search: comparisons

Construct $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$ incrementally (checking for one matrix $N \in D(X)(\mathbf{0},\mathbf{1})$ at a time), starting with matrix $\mathbf{1}$:

1. $N_{i+1} \not\leq M \in \mathcal{C}^i(X) \Rightarrow M \in \mathcal{C}^{i+1}(X)$;
2. $N_{i+1} \leq M \Rightarrow M$ is replaced by n matrices M^j with one additional 0. Example: $X = \vec{I}^n \setminus \vec{J}^n$.

Minimal transversals in a hypergraph

- A matrix in $D(X)(\mathbf{0},\mathbf{1})$ describes a hyperedge on the vertex set $[1:l] \times [1:n]$; $D(X)(\mathbf{0},\mathbf{1})$ describes a hypergraph.
- A transversal in a hypergraph is a vertex set that has non-empty intersection with each hyperedge.
- Complements of minimal transversals correpond to matrices in $\mathcal{C}_{\max}(\mathbf{0},\mathbf{1})$ – algorithms well-developed.

More general linear semaphore state spaces

- More general semaphores (intersection with the boundary of I^n allowed)
- n dining philosophers: Trace space has $2^n - 2$ components
- Different end points: $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ and iterative calculations
- End complexes rather than end points (allowing processes not to respond..., Herlihy & Cie)
- Same technique, modification of definition and calculation of $\mathcal{C}(X)(-,-)$, $D(X)(-,-)$ etc. ; cf preprint, submitted.

State space components

New light on definition and determination of components of model space X.

Extensions

2a. Semaphores corresponding to non-linear programs:

Path spaces in product of digraphs

Products of digraphs instead of $\vec{I}^n$:
$\Gamma = \prod_{j=1}^n \Gamma_j$, state space $X = \Gamma \setminus F$,
F a product of generalized hyperrectangles R^i.

- $\vec{P}(\Gamma)(\mathbf{x}, \mathbf{y}) = \prod \vec{P}(\Gamma_j)(x_j, y_j)$ – homotopy discrete!

Pullback to linear situation

Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$ by(regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j, y_j)$ – an interleaving.
The map $c : \vec{I}^n \to \Gamma, c(t_1, \dots, t_n) = (c_1(t_1), \dots, c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0}, \mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

Homotopy types of interleaving components

Pull back F via c:
$\bar{X}=\vec{I}^n\setminus\bar{F}, \bar{F}=\bigcup\bar{R}^i, \bar{R}^i=c^{-1}(R^i)$ – honest hyperrectangles!
$i_X:\vec{P}(X)\hookrightarrow\vec{P}(\Gamma)$.
Given a component $C\subset\vec{P}(\Gamma)(\mathbf{x},\mathbf{y})$.
The d-map $c:\bar{X}\to X$ induces a homeomorphism
$c\circ:\vec{P}(\bar{X})(\mathbf{0},\mathbf{1})\to i_X^{-1}(C)\subset\vec{P}(X)(\mathbf{x},\mathbf{y})$.

- C “lifts to X” $\Leftrightarrow\vec{P}(\bar{X})(\mathbf{0},\mathbf{1})\neq\emptyset$; if so:
- Analyse $i_X^{-1}(C)$ via $\vec{P}(\bar{X})(\mathbf{0},\mathbf{1})$.
- Exploit relations between various components.

Special case: $\Gamma=(S^1)^n$ – a torus

State space: A torus with rectangular holes in F:
Investigated by Fajstrup, Goubault, Mimram etal.:
Analyse by language on the alphabet $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ of alive matrices for a one-fold delooping of $\Gamma\setminus F$.

HDA: Directed pre-cubical complex

Higher Dimensional Automaton: Pre-cubical complex – like simplicial complex but with cubes as building blocks – with preferred diretions.
Geometric realization X with d-space structure.

Branch points and branch cubes

These complexes have branch points and branch cells – more than one maximal cell with same lower corner vertex.
At branch points, one can cut up a cubical complex in simpler pieces.
Trouble: Simpler pieces may have higher order branch points.

Non-branching complexes

Start with complex without directed loops: After finally many iterations: Subcomplex Y without branch points.

Theorem

$\vec{P}(Y)(\mathbf{x}_0, \mathbf{x}_1)$ is empty or contractible.

Proof.

Such a subcomplex has a preferred diagonal flow and a contraction from path space to the flow line from start to end. □

Results in a (complicated) finite category $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ on subsets of (iterated) branch cells.

Theorem

$\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of that category.

Delooping HDAs

A pre-cubical complex comes with an L_1-length 1-form $\omega = dx_1 + \cdots + dx_n$ on every n-cube.
Integration: L_1-length on rectifiable paths, homotopy invariant.
Defines $l : P(X)(x_0, x_1) \to \mathbf{R}$ and $l_\sharp : \pi_1(X) \to \mathbf{R}$ with kernel K.
The (usual) covering $\tilde{X} \downarrow X$ with $\pi_1(\tilde{X}) = K$ is a directed pre-cubical complex without directed loops.

Theorem (Decomposition theorem)

For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n\in\mathbf{Z}} \vec{P}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$[a].

[a] in the fibres over $\mathbf{x}_0, \mathbf{x}_1$

To conclude

- From a (rather compact) state space model to a finite dimensional trace space model.
- Calculations of invariants (Betti numbers) possible even for quite large state spaces.
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors) – linearly.
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  (Connection to the work of Herlihy and Rajsbaum)

Want to know more?
Thank you!

- Eric Goubault's talk this afternoon!

References

- MR, Simplicial models for trace spaces, AGT **10** (2010), 1683 – 1714.
- MR, Execution spaces for simple higher dimensional automata, Aalborg University Research Report R-2010-14; submitted
- MR, Simplicial models for trace spaces II: General HDA, Draft.
- Fajstrup, Trace spaces of directed tori with rectangular holes, Aalborg University Research Report R-2011-08.
- Fajstrup etal., Trace Spaces: an efficient new technique for State-Space Reduction, submitted.
- Rick Jardine, Path categories and resolutions, Homology, Homotopy Appl. **12** (2010), 231 – 244.

Thank you for your attention!