Trace spaces: Organization, Calculations, Applications

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Outline

1. Motivations, mainly from Concurrency Theory
2. Directed topology: algebraic topology with a twist
3. Trace spaces: definitions, calculations via classical algebraic topology
4. Categorical organization of invariants

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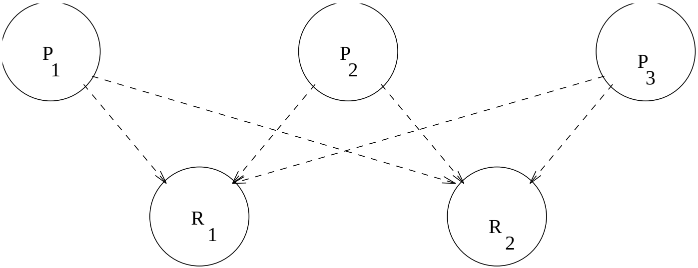
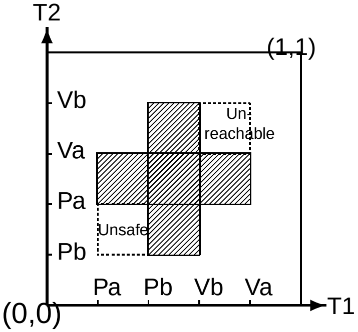
Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.
Semaphores!
Semantics: A processor has to lock a resource and relinquish the lock later on!
Description/abstraction $P_i : \ldots PR_j \ldots VR_j \ldots$ (Dijkstra)

Schedules in ”progress graphs”

The Swiss flag example



PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

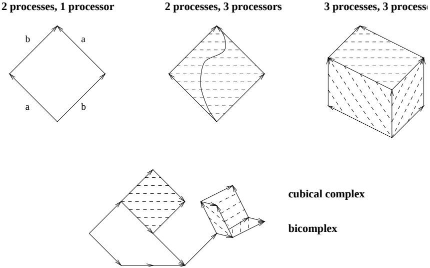
Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

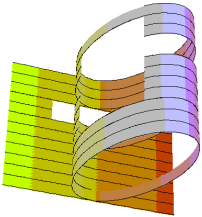
Deadlocks, unsafe and unreachable regions may occur.

Higher dimensional automata

seen as (geometric realizations of) cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...





Squares/cubes/hypercubes are filled in iff actions on boundary are independent.

Higher dimensional automata are (pre)-cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps
- additionally: preferred directions – not all paths allowable.

Discrete versus continuous models

How to handle the state-space explosion problem?

The state space explosion problem for discrete models for concurrency (transition graph models): The number of states (and the number of possible schedules) grows exponentially in the number of processors and/or the length of programs.

Need clever ways to find out which of the schedules yield equivalent results for general reasons – e.g., to check for correctness.

Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!

Analogy: Continuous physics as an approximation to (discrete) quantum physics.

A general framework for directed topology

The twist: d-spaces, M. Grandis (03)

X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$ a space of d-paths (CO-topology; "directed" paths $\leftrightarrow$ executions) satisfying

- { constant paths } $\subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.
Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t,\ \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

D-maps, Dihomotopy, d-homotopy

A d-map $f : X \to Y$ is a continuous map satisfying

- $f(\vec{P}(X)) \subseteq \vec{P}(Y)$.

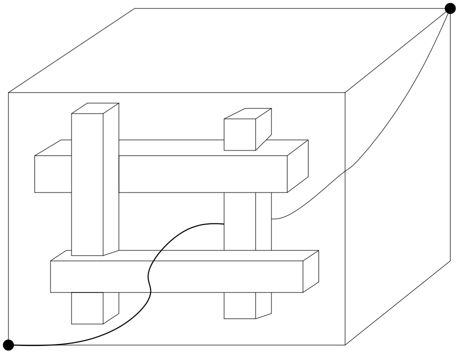
special case: $\vec{P}(I) = \{\sigma \in I^I | \sigma$ nondecreasing reparametrization$\}, \vec{I} = (I, \vec{P}(I))$.
Then $\vec{P}(X) =$ space of d-maps from $\vec{I}$ to X.

- Dihomotopy $H : X \times I \to Y$, every H_t a d-map
- elementary d-homotopy = d-map $H : X \times \vec{I} \to Y$ – $H_0 = f \xrightarrow{H} g = H_1$
- d-homotopy: symmetric and transitive closure (”zig-zag”)

L. Fajstrup, 05: In cubical models (for concurrency, e.g., HDAs), the two notions agree for d-paths ($X = \vec{I}$). In general, they do not.

Dihomotopy is finer than homotopy with fixed endpoints

Example: Two wedges in the forbidden region

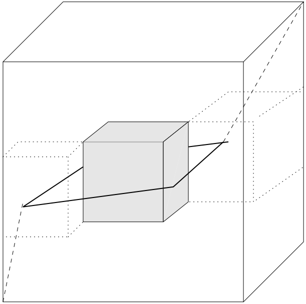


All dipaths from minimum to maximum are homotopic.
A dipath through the “hole” is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

Ordinary topology: Path space = loop space (within each path component).
A loop space is an H-space with concatenation, inversion, cancellation.



"Birth and death" of d-homotopy classes

Directed topology: Loops do not tell much; concatenation ok, cancellation not!
Replace group structure by category structures!

X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$ ("same oriented trace").
(Fahrenberg-R., 07): Reparametrization equivalence is an equivalence relation (transitivity).
$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative.
A d-map $f : X \to Y$ induces a functor $\vec{T}(f) : \vec{T}(X) \to \vec{T}(Y)$.

The two main objectives

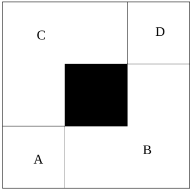
- Investigation/calculation of the homotopy type of trace spaces $\vec{T}(X)(x,y)$ for relevant d-spaces X
- Investigation of topology change under

$$\vec{T}(X)(x',y) \stackrel{\sigma^*_{x'x}}{\leftarrow} \vec{T}(X)(x,y) \stackrel{\sigma_{yy'*}}{\longrightarrow} \vec{T}(X)(x,y')$$

Categorical organization

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x, y)$ depend on (x, y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
Problem: This “compression” works only for loopfree categories (d-spaces)

Preorder categories

Getting organised with indexing categories

A d-space structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: (end point) pairs $(x,y), x \preceq y$
- Morphisms:
  $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$x' \rightrightarrows x \xrightarrow{\preceq} y \rightrightarrows y'$$

- Composition: by pairwise contra-, resp. covariant concatenation.

A d-map $f : X \to Y$ induces a functor $\vec{D}(f) : \vec{D}(X) \to \vec{D}(Y)$.

The trace space functor

Preorder categories organise the trace spaces

The preorder category organises X via the
trace space functor $\vec{T}^X : \vec{D}(X) \to Top$

- $\vec{T}^X(x,y) := \vec{T}(X)(x,y)$
- $\vec{T}^X(\sigma_x, \sigma_y) : \quad \vec{T}(X)(x,y) \longrightarrow \vec{T}(X)(x',y')$

$$[\sigma] \longmapsto [\sigma_x * \sigma * \sigma_y]$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms
$\vec{D}_\pi(X)((x,y),(x',y')) := \vec{\pi}_1(X)(x',x) \times \vec{\pi}_1(X)(y,y')$
and trace space functor $\vec{T}^X_\pi : \vec{D}_\pi(X) \to Ho - Top$ (with homotopy classes as morphisms).

D-homology $\vec{H}_*$

For every d-space X, there are homology functors

$$\vec{H}_{*+1}(X) = H_* \circ \vec{T}_\pi^X : \vec{D}_\pi(X) \to Ab,\ (x,y) \mapsto H_*(\vec{T}(X)(x,y))$$

capturing homology of all relevant d-path spaces in X
and the effects of the concatenation structure maps.
A d-map $f : X \to Y$ induces a natural transformation $\vec{H}_{*+1}(f)$ from $\vec{H}_{*+1}(X)$ to $\vec{H}_{*+1}(Y)$.
Similarly for other algebraic topological functors; a bit more complicated for homotopy groups: base points!

- How much does (the homotopy type of) $\vec{T}^X(x,y)$ depend on (small) changes of x,y?
- Which concatenation maps $\vec{T}^X(\sigma_x,\sigma_y):\vec{T}^X(x,y)\to\vec{T}^X(x',y'),\ [\sigma]\mapsto[\sigma_x*\sigma*\sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there "components" with (homotopically/homologically) stable dipath spaces (between them)? Are there borders ("walls") at which changes occur?

Topology of trace spaces for a cubical complex X

l^1 "arc length" parametrization: on each cube, arc length is the l^1-distance of end-points. Additive continuation $\rightsquigarrow$ subspace of arc-length parametrized d-paths $\vec{P}_n(X) \subset \vec{P}(X)$.
D-homotopic paths in $\vec{P}_n(X)(x,y)$ have the same arc length!
The spaces $\vec{P}_n(X)$ and $\vec{T}(X)$ are homeomorphic, $\vec{P}(X)$ is homotopy equivalent to both.

Theorem

X a pre-cubical set; $x, y \in X$. Then $\vec{T}(X)(x,y)$

- *is metrizable, locally contractible and locally compact*[1]*.*
- *has the homotopy type of a CW-complex.*[2]

First examples

I^n the unit cube, ∂I^n its boundary.

- $\vec{T}(I^n; \mathbf{x}, \mathbf{y})$ is contractible for all $x \preceq y \in I^n$;
- $\vec{T}(\partial I^n; \mathbf{0}, \mathbf{1})$ is homotopy equivalent to S^{n-2}.

[1] MR, Trace spaces in pre-cubical complexes, Draft

Let $L \subset X$ denote a (properly chosen) subspace. Investigate the concatenation map

$c_L : \vec{T}(x_0, L) \times_L \vec{T}(L, x_1) \to \vec{T}(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$

onto? fibres? Topology of the pieces? Generalization:

$L_1, \cdots, L_k$ a sequence of (properly chosen) subspaces.

Investigate the concatenation map

$\vec{T}(X)(x_0, L_1) \times_{L_1} \cdots \times_{L_j} \vec{T}(X)(L_j, L_{j+1}) \times_{L_{j+1}} \cdots \times_{L_k} \vec{T}(X)(L_n, x_1)$.

onto? fibres? Topology of the pieces?

Tool : The Vietoris-Begle mapping theorem

Stephen Smale's version for homotopy groups

What does a surjective map $p : X \to Y$ with highly connected fibres $p^{-1}(y), y \in Y$, tell about invariants of X, Y?

The Vietoris-Begle mapping theorem compares the Alexander-Spanier cohomology groups of X, Y.

Stephen Smale, *A Vietoris Mapping Theorem for Homotopy*, Proc. Amer. Math. Soc. **8** (1957), no. 3, 604 – 610:

Theorem

Let $f : X \to Y$ denote a proper surjective map between connected locally compact separable metric spaces. Let moreover X be locally n-connected, and for each $y \in Y$, let $f^{-1}(y)$ be locally $(n-1)$-connected and $(n-1)$-connected. Then

1. *Y is locally n-connected, and*
2. *$f_\# : \pi_r(X) \to \pi_r(Y)$ is an isomorphism for all $0 \leq r \leq n-1$ and onto for $r = n$.*

An important special case

All fibres contractible and locally contractible

Corollary

Let $f : X \to Y$ denote a proper surjective map between locally compact separable metric spaces. Let moreover X be locally contractible, and for each $y \in Y$, let $f^{-1}(y)$ be contractible and locally contractible. Then

1. *Y is locally contractible, and*
2. *f is a weak homotopy equivalence.*

Definition

A subset $A \subseteq X$ of a d-space X is called

- d-convex if $[x_0, x_1] = \{p(t) \mid p \in \vec{P}(X)(x_0, x_1), t \in I\} \subseteq A$; in particular, $p^{-1}(A)$ is either an interval or empty for all $p \in \vec{P}(X)$;
- unavoidable from $B \subset X$ to $C \subset X$ if $\vec{P}(X \setminus A)(B, C) = \emptyset$.

Theorem

Let X be a nice d-space, e.g., the geometric realization of a pre-cubical complex. Let $x_0, x_1 \in X$, $L \subset X$ d-convex and unavoidable from x_0 to x_1.
If $\vec{T}(X)(x_0, L)$ and $\vec{T}(X)(L, x_1)$ are locally contractible, then the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$
is a weak (?) homotopy equivalence.

An important special case

Corollary

If $\vec{T}(X)(x_0, l)$ and $\vec{T}(X)(l, x_1)$ are contractible and locally contractible for every $l \in L$, then $\vec{T}(X)(x_0, x_1)$ is weakly (?) homotopy equivalent to L.

Proof.

The fibre over $l \in L$ of the "mid point" map
$m : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to L$ is
$m^{-1}(l) = \vec{T}(X)(x_0, l) \times \vec{T}(X)(l, x_1)$. □

Example

$X = \partial I^n = \{\mathbf{x} \in I^n \mid \exists i : x_i = 0 \vee x_i = 1\} \simeq S^{n-1}$
$L = \partial_{\pm} I^n = \{\mathbf{x} \in I^n \mid \exists i, j : x_i = 0, x_j = 1\} \simeq S^{n-2}$
Then $\vec{T}(\partial I^n)(\mathbf{0}, \mathbf{1})$ is weakly homotopy equivalent[3] to S^{n-2}.

[3] in fact homotopy equivalent

Key points in the proof of Theorem

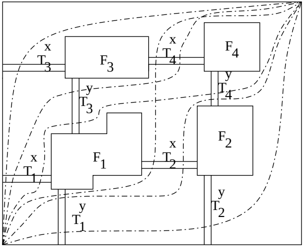
- Check the topological conditions (separable metric space, locally compact, locally contractible, properness) for trace spaces in nice d-spaces[4].
- Surjectivity corresponds to unavoidability.
- D-convexity ensures that every fibre is an interval, hence contractible.
- Under which conditions to L can Milnor's proof be adapted to get an actual homotopy equivalence?

[4] MR, Trace spaces in pre-cubical complexes, manuscript

Given a collection $\mathcal{L}$ of $m+1$ (finitely many) disjoint subsets $L_i \subset X$ with $L_0 = \{x_0\}, L_m = \{x_1\}$.
A d-path in X is called prime with respect to $\mathcal{L}$ if there are $L_i, L_j \in \mathcal{L}$ and $a, b \in I$ such that
$p^{-1}(L_i) = [0, a], p^{-1}(L_j) = [b, 1]$ and $p^{-1}(L_k) = \emptyset$ for $k \neq i, j$.
Let $\vec{P}^{\mathcal{L}}(X) \subset \vec{P}(X)$ denote the subspace of all d-paths that are prime with respect to $\mathcal{L}$.
The collection $\mathcal{L}$ is called unavoidable from x_0 to x_1 if

- every d-path $p \in \vec{P}(X)(x_0, x_1)$ can be decomposed into pieces that are prime with respect to $\mathcal{L}$;
- every d-path $q \in \vec{P}^{\mathcal{L}}(X)(L_i, L_j)$ that is d-homotopic (rel L_i, L_j) to a prime d-path is prime itself.

A sequence $(0, i_1, \ldots, i_n, m)$ is $\mathcal{L}$-admissible if
$\vec{P}^{\mathcal{L}}(X)(L_{i_j}, L_{i_{j+1}}) \neq \emptyset, 0 \leq j \leq n$.



Theorem

Let $\mathcal{L}$ denote a collection of finitely many disjoint subsets in X that is unavoidable from x_0 to x_1. Then $\vec{T}(X)(x_0, x_1)$ is weakly homotopy equivalent to the disjoint union over all $\mathcal{L}$-admissible sequences $(0, i_1, \ldots, i_n, 1)$ of spaces

$$\vec{T}^{\mathcal{L}}(X)(x_0, L_{i_1}) \times_{L_{i_1}} \cdots \times_{L_{i_j}} \vec{T}^{\mathcal{L}}(X)(L_{i_j}, L_{i_{j+1}}) \times_{L_{i_{j+1}}} \cdots \times_{L_{i_n}} \vec{T}^{\mathcal{L}}(X)(L_{i_n}, x_1).$$

Proof.

Apply Smale's Vietoris theorem to the concatenation map into $\vec{T}(X)(x_0, x_1)$.

- Unavoidability ensures surjectivity.
- Since the pieces are prime, every fibre is a product of intervals, hence contractible.

An important special case

Reachability. For a given collection $\mathcal{L}$ of finitely many disjoint subsets in X that is unavoidable from x_0 to x_1, let $R^{\mathcal{L}}(L_i, L_j) = \{(x_i, x_j) \in L_i \times L_j \mid \vec{P}^{\mathcal{L}}(x_i, x_j) \neq \emptyset\} \subset X \times X$.

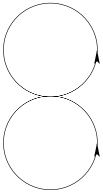
Corollary

If, moreover, for all i, j, $(x_i, x_j) \in R^{\mathcal{L}}(L_i, L_j)$ the path spaces $\vec{T}^{\mathcal{L}}(X)(x_i, x_j)$ are contractible and locally contractible, then $\vec{T}(X)(x_0, x_1)$ is weakly homotopy equivalent to the disjoint union over all $\mathcal{L}$-admissible sequences $(0, i_1, \dots, i_n, 1)$ of spaces
$R^{\mathcal{L}}(x_0, L_{i_1}) \times_{L_{i_1}} \cdots \times_{L_{i_j}} R^{\mathcal{L}}(L_{i_j}, L_{i_{j+1}}) \times_{L_{i_{j+1}}} \cdots \times_{L_{i_n}} R^{\mathcal{L}}(L_{i_n}, x_1) \subset X^{n+1}$.

The latter space consists of sequences of mutually reachable points in the given layers.

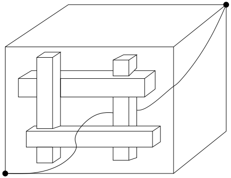
Examples

1.



A wedge of two directed circles $X = \vec{S}^1 \vee_{x_0} \vec{S}^1$:
$\vec{T}(X)(x_0, x_0) \simeq \{1, 2\}^*$.
(Choose $L_i = \{x_i\}, i = 1, 2$ with $x_i \neq x_0$ on the two branches).

2.



$Y =$ cube with two wedges deleted:
$\vec{T}(Y)(\mathbf{0}, \mathbf{1}) \simeq * \sqcup (S^1 \vee S^1)$.
(L_i two vertical cuts through the wedges; product is homotopy equivalent to torus; reachability $\rightsquigarrow$ two components, one of which is contractible, the other a thickening of $S^1 \vee S^1 \subset S^1 \times S^1$.)

Let X denote the geometric realization of a finite pre-cubical complex ($\square$-set) M, i.e., $X = \coprod (M_n \times \vec{I}^n)_{/\simeq}$.
X consists of “cells” e_α homeomorphic to I^{n_α}. A cell is called maximal if it is not in the image of a boundary map $\partial^\pm$.
The d-path structure $\vec{P}(X)$ is inherited from the $\vec{P}(\vec{I}^n)$ by “pasting”.

Definition

$p \in \vec{P}(X)$ is called PL if: $p(t) \in e_\alpha$ for $t \in J \subseteq I \Rightarrow p_{|J}$ linear[5].
$\vec{P}_{PL}(X), \vec{T}_{PL}(X)$: subspaces of linear d-paths and traces.

Theorem

For all $x_0, x_1 \in X$, the inclusion $\vec{T}_{PL}(X)(x_0, x_1) \hookrightarrow \vec{T}(X)(x_0, x_1)$ is a weak homotopy equivalence.

[5] and close-up on boundaries

A sequence of cells $(e_{\alpha_0}, \dots, e_{\alpha_n})$ in X is called a chain from x_0 to x_1 if every of the cells is maximal, if $x_0 \in e_{\alpha_0}, x_1 \in e_{\alpha_n}$ and if $\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}} \neq \emptyset$ for $0 \le i < n$.

Let $C(X)(x_0, x_1)$ denote the set of chains in X from x_0 to x_1.

Apply Smale's Vietoris theorem to the concatenation map:

$\bigcup_{c \in C(X)(x_0,x_1)} \vec{T}(X)(x_0, \partial^+ e_{\alpha_0}) \times_{\partial^+ e_{\alpha_0} \cap \partial^- e_{\alpha_1}} \cdots \times_{\partial^+ e_{\alpha_{i-1}} \cap \partial^- e_{\alpha_i}}$
$\vec{T}(X)(\partial^- e_{\alpha_i}, \partial^+ e_{\alpha_i}) \times_{\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}}} \cdots \times_{\partial^+ e_{\alpha_{n-1}} \cap \partial^- e_{\alpha_n}}$
$\vec{T}(X)(\partial^- e_{\alpha_n}, x_1) \to \vec{T}(X)(x_0, x_1)$.

This map is a weak homotopy equivalence: It is surjective; the fibres are products of intervals, hence contractible.

Paste homotopy equivalences on factors $\vec{T}_{PL}(X)(\partial^- e_{\alpha_i}, \partial^+ e_{\alpha_i}) \hookrightarrow \vec{T}(X)(\partial^- e_{\alpha_i}, \partial^+ e_{\alpha_i})$.

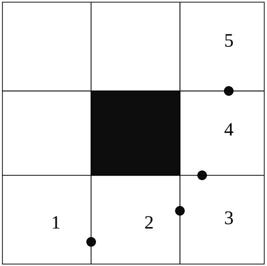
Definition

A maximal cube path in a pre-cubical set is a sequence $(e_{\alpha_1}, \dots, e_{\alpha_k})$ of maximal cells such that $\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}} \neq \emptyset$.

The *PL*-traces within a given maximal cube path $(e_{\alpha_1}, \dots, e_{\alpha_k})$ correspond to sequences in $\{(y_1, \dots, y_{k-1}) \in \prod_{i=1}^{k-1}(\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}}) \subset X^k \mid \vec{P}(e_{\alpha_i})(y_{i-1}, y_i) \neq \emptyset, 1 < i < k\}$.
This set carries a natural structure as a product of simplices $\prod \Delta^{j_k}$.

Subsimplices and their products: Some coordinates of d-paths are minimal, maximal or fixed within one or several cells.
The space $\vec{T}_{PL}(X)$ carries thus the structure of a prodsimplicial complex $\rightsquigarrow$ possibilities for inductive calculations.

Simple examples

1. 

   Two maximal cube paths from **0** to **1**, each of them contributing $\Delta^2 \times \Delta^2$. Empty intersection.
   $\vec{T}_{PL}(X)(\mathbf{0},\mathbf{1}) \simeq (\Delta^2 \times \Delta^2) \sqcup (\Delta^2 \times \Delta^2)$.
2. $X = \partial \vec{I}^n$. Maximal cube paths from **0** to **1** have length 2. Every PL-d-path is determined by an element of $\partial_{\pm} \vec{I}^n \simeq S^{n-2}$.

Future work

on the algebraic topology of trace spaces

- Is there an automatic way to place consecutive “diagonal cut” layers in complexes corresponding to PV-programs that allow to compare path spaces to subspaces of the products of these layers?
- PL-d-paths come in “rounds” corresponding to the sums of dimensions of the cells they enter. This gives hope for inductive calculations (as in the work of Herlihy, Rajsbaum and others) in distributed computing.
- Explore the combinatorial algebraic topology of the trace spaces
  - with fixed end points and
  - what happens under variations of end points.
- Make this analysis useful for the determination of components (extend the work of Fajstrup, Goubault, Haucourt, MR)

Examples of component categories

Example 1: No nontrivial d-loops

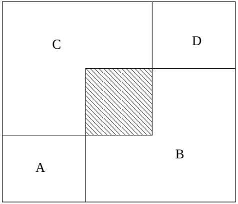
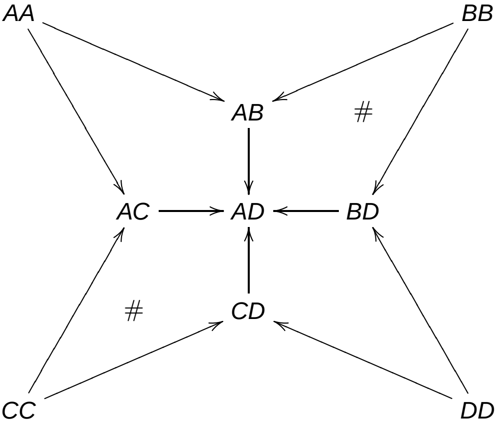


Figure: Deleted square with component category



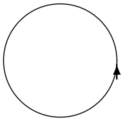
Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD \subseteq X \times X$.

$\#$: diagram commutes.

Examples of component categories

Example 2: Oriented circle

$X \quad = \quad \vec{S}^1$



oriented circle

$\vec{T}(\vec{S}^1)(x,y) \simeq \mathbf{N}_0$.

$$\mathcal{C} : \Delta \underset{b}{\overset{a}{\rightleftarrows}} \bar{\Delta}$$

Δ the diagonal, $\bar{\Delta}$ its complement. $\mathcal{C}$ is the free category generated by a, b.

- Remark that the components are no longer products!
- In order to get a discrete component category, it is essential to use an indexing category taking care of pairs (source, target).

Tool: Homotopy flows

in particular: Automorphic homotopy flows

A d-map $H : X \times \vec{I} \to X$ is called a (f/p) homotopy flow if

$$\text{future} \quad H_0 = id_X \xrightarrow{H} f = H_1$$

$$\text{past} \quad H_0 = g \xrightarrow{H} id_X = H_1$$

H_t is not a homeomorphism, in general; the flow is irreversible.
H and f are called

automorphic if $\vec{T}(H_t) : \vec{T}(X)(x, y) \to \vec{T}(X)(H_t x, H_t y)$ is a homotopy equivalence for all $x \preceq y, t \in I$.

Automorphisms are closed under composition – concatenation of homotopy flows!
$Aut_+(X), Aut_-(X)$ monoids of automorphisms.
Variations: $\vec{T}(H_t)$ induces isomorphisms on homology groups, homotopy groups....

Compression: Generalized congruences and quotient categories

Bednarczyk, Borzyszkowski, Pawlowski, TAC 1999

How to identify morphisms in a category between different objects in an organised manner?

Start with an equivalence relation $\simeq$ on the objects.

A generalized congruence is an equivalence relation on non-empty sequences $\varphi = (f_1 \dots f_n)$ of morphisms with $cod(f_i) \simeq dom(f_{i+1})$ ($\simeq$-paths) satisfying

1. $\varphi \simeq \psi \Rightarrow dom(\varphi) \simeq dom(\psi), codom(\varphi) \simeq codom(\psi)$
2. $a \simeq b \Rightarrow id_a \simeq id_b$
3. $\varphi_1 \simeq \psi_1$, $\varphi_2 \simeq \psi_2$, $cod(\varphi_1) \simeq dom(\varphi_2) \Rightarrow \varphi_2\varphi_1 \simeq \psi_2\psi_1$
4. $cod(f) = dom(g) \Rightarrow f \circ g \simeq (fg)$

Quotient category $\mathcal{C}/\simeq$: Equivalence classes of objects and of $\simeq$-paths; composition: $[\varphi] \circ [\psi] = [\varphi\psi]$.

Automorphic homotopy flows give rise to generalized congruences

Let X be a d-space and $Aut_{\pm}(X)$ the monoid of all (future/past) automorphisms.
"Flow lines" are used to identify objects (pairs of points) and morphisms (classes of d-paths) in an organized manner.
$Aut_{\pm}(X)$ gives rise to a generalized congruence on the (homotopy) preorder category $\vec{D}_{\pi}(X)$ as the symmetric and transitive congruence closure of:

Congruences and component categories

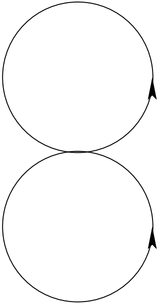
- $f_+ : (x,y) \overset{\simeq}{\leftrightarrow} (x',y') : f_-, \qquad f_\pm \in Aut_\pm(X)$
- $$(x,y) \overset{(\sigma_1,\sigma_2)}{\longrightarrow} (u,v) \simeq (x',y') \overset{(\tau_1,\tau_2)}{\longrightarrow} (u',v'),$$
  $f_+ : (x,y,u,v) \leftrightarrow (x',y',u',v') : f_-, \qquad f_\pm \in Aut_\pm(X)$, and
  $$\begin{array}{ccc} \vec{T}(X)(x',y') & \overset{(\tau_1,\tau_2)}{\longrightarrow} & \vec{T}(X)(u',v') \\ \vec{T}(f_+)\uparrow\downarrow\vec{T}(f_-) & & \vec{T}(f_+)\uparrow\downarrow\vec{T}(f_-) \\ \vec{T}(X)(x,y) & \overset{(\sigma_1,\sigma_2)}{\longrightarrow} & \vec{T}(X)(u,v) \end{array}$$
  commutes (up to ...).
- $(x,y) \overset{(c_x,H_y)}{\longrightarrow} (x,fy) \simeq (fx,fy) \overset{(H_x,c_{fy})}{\longrightarrow} (x,fy), \; H : id_X \to f.$
  Likewise for $H : g \to id_X$.

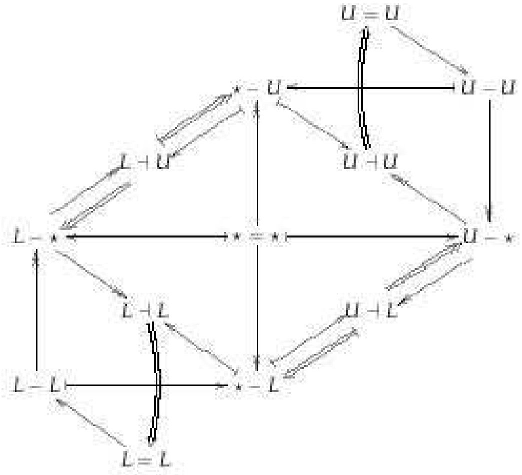
The component category $\vec{D}_\pi(X)/\simeq$ identifies pairs of points on the same "homotopy flow line" and (chains of) morphisms.

Examples of component categories 1

Example 3: The component category of a wedge of two oriented circles

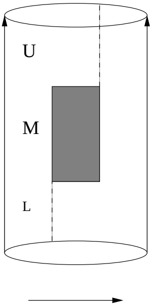
$X = \vec{S}^1 \vee \vec{S}^1$
$\vec{T}(X)(x,y) \simeq$
$\mathbf{N}_0 * \mathbf{N}_0$

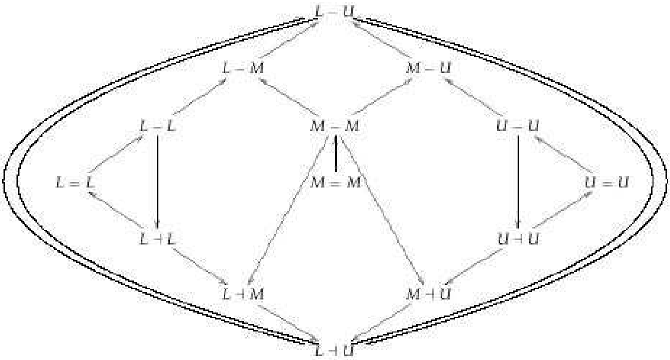




Examples of component categories

Example 4: The component category of an oriented cylinder with a deleted rectangle





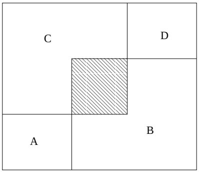
Dihomotopy equivalence – a naive definition

Definition

A d-map $f : X \to Y$ is a dihomotopy equivalence if there exists a d-map $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

But this does not imply an obvious property wanted for:
A dihomotopy equivalence $f : X \to Y$ should induce (ordinary) homotopy equivalences

$$\vec{T}(f) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy)!$$



A map d-homotopic to the identity does not preserve homotopy types of trace spaces? Need to be more restrictive!

Definition

A d-map $f : X \to Y$ is called a future dihomotopy equivalence if there are maps $f_+ : X \to Y, g_+ : Y \to X$ with $f \to f_+$ and automorphic homotopy flows $id_X \to g_+ \circ f_+, id_Y \to f_+ \circ g_+$.
Property of dihomotopy class!

likewise: past dihomotopy equivalence $f_- \to f, g_- \to g$
dihomotopy equivalence = both future and past dhe
(g_-, g_+ are then d-homotopic).

Theorem

A (future/past) d-homotopy equivalence $f : X \to Y$ induces homotopy equivalences

$$\vec{T}(f)(x,y) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy) \text{ for all } x \preceq y.$$

Moreover: (All sorts of) Dihomotopy equivalences are closed under composition.

Concluding remarks

- Component categories contain the essential information given by (algebraic topological invariants of) path spaces
- Compression via component categories is an antidote to the state space explosion problem
- Some of the ideas (for the fundamental category) are implemented and have been tested for huge industrial software from EDF (Éric Goubault & Co., CEA)
- Dihomotopy equivalence: Definition uses automorphic homotopy flows to ensure homotopy equivalences

$$\vec{T}(f)(x,y) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy) \text{ for all } x \preceq y.$$

- Much more theoretical and practical work remains to be done!