Supplement Material for "Nonparametric Estimation of the Pair Correlation Function of Replicated Inhomogeneous Point processes"

Ganggang Xu, Chong Zhao, Abdollah Jalilian, Rasmus Waagepetersen, Jingfei Zhang, Yongtao Guan

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Abstract

This document contains all technical proofs for the paper 'Nonparametric Estimation of the Pair Correlation Function of Replicated Inhomogeneous Point processes'.

KEY WORDS: Confidence band; Estimating equations; Local polynomial estimator; Nonparametric estimation; Orthogonal series estimator; Replicated point patterns.

Ganggang Xu (Email: gangxu@bus.miami.edu) and Jingfei Zhang (Email: ezhang@bus.miami.edu) are Assistant Professors, Yongtao Guan is Professor (Email: yguan@bus.miami.edu), Department of Management Science, University of Miami, Coral Gables, FL 33124. Chong Zhao (Email: zhaochongyct@gmail.com) is Statistical Analyst, Equifax, Alpharetta, GA 30005; Abdollah Jalilian is Assistant Professor (Email: jalilian@razi.ac.ir), Department of Statistics, Razi University, Bagh-e-Abrisham, Kermanshah, Iran, 67144. Rasmus Waagepetersen is Professor (Email: rw@math.aau.dk), Department of Mathematical Sciences, Aalborg University, DK-9220. For correspondence, please contact Ganggang Xu.

1 Proof of Lemma 1

Proof. When $\lambda(\mathbf{s}) \equiv \lambda$ and the observation window is $D_n = [0, T_n] \subset \mathbb{R}$, we have that

$$
\begin{aligned}
\mathbf{Q}_{n,h}^{(k)}(r) &= \lambda^2 \frac{1}{|D_n|h} \int_{-T_n}^{T_n} |D_n \cap (D_n - s)| \left[K\left(\frac{|s|-r}{h}\right)\right]^k \boldsymbol{A}_h(|s|-r)\boldsymbol{A}_h^T(|s|-r)\mathrm{d}s \\
&= \lambda^2 \frac{1}{T_n h} \int_{-T_n}^{T_n} (T_n - |s|) \left[K\left(\frac{|s|-r}{h}\right)\right]^k \boldsymbol{A}_h(|s|-r)\boldsymbol{A}_h^T(|s|-r)\mathrm{d}s \\
&= \lambda^2 \frac{2}{T_n h} \int_{0}^{T_n} (T_n - s) \left[K\left(\frac{s-r}{h}\right)\right]^k \boldsymbol{A}_h(s-r)\boldsymbol{A}_h^T(s-r)\mathrm{d}s \\
&= \lambda^2 \frac{2}{T_n h} \int_{-r}^{T_n - r} (T_n - s - r) \left[K\left(\frac{s}{h}\right)\right]^k \boldsymbol{A}_h(s)\boldsymbol{A}_h^T(s)\mathrm{d}s \\
&= \lambda^2 \frac{2}{T_n} \int_{-r/h}^{(T_n - r)/h} (T_n - sh - r) \left[K(s)\right]^k \boldsymbol{A}_1(s)\boldsymbol{A}_1^T(s)\mathrm{d}s.
\end{aligned}
$$

If we use the uniform kernel $K(x) = \frac{1}{2}I(-1 \le x \le 1)$, the above equation can be further simplified as

$$
\begin{aligned}
\mathbf{Q}_{n,h}^{(k)}(r) &= \frac{\lambda^2}{2^{k-1}} \frac{1}{T_n} \int_{\max(-r/h,-1)}^{\min[(T_n-r)/h,1]} (T_n - sh - r)\boldsymbol{A}_1(s)\boldsymbol{A}_1^T(s)\mathrm{d}s \\
&= \frac{\lambda^2}{2^{k-1}} \left(1 - \frac{r}{T_n}\right) \int_{\max(-r/h,-1)}^{\min[(T_n-r)/h,1]} \boldsymbol{A}_1(s)\boldsymbol{A}_1^T(s)\mathrm{d}s \\
&\quad - \frac{\lambda^2}{2^{k-1}} \frac{h}{T_n} \int_{\max(-r/h,-1)}^{\min[(T_n-r)/h,1]} s\boldsymbol{A}_1(s)\boldsymbol{A}_1^T(s)\mathrm{d}s \\
&= \frac{\lambda^2}{2^{k-1}} \left(1 - \frac{r}{T_n}\right) \mathbf{B}_1(r) - \frac{\lambda^2}{2^{k-1}} \frac{h}{T_n} \mathbf{B}_2(r),
\end{aligned}
$$

where $\mathbf{B}_1(r)$ is a $(p+1)\times(p+1)$ matrix whose (i,j)th entry is $\frac{1}{i+j-1}(q_{up}^{i+j-1} - q_{low}^{i+j-1})$ and $\mathbf{B}_2(r)$ is a $(p+1)\times(p+1)$ matrix whose (i,j)th entry is $\frac{1}{i+j}(q_{up}^{i+j} - q_{low}^{i+j})$, with $q_{low} = \max(-r/h, -1)$ and $q_{up} = \min[(T_n - r)/h, 1]$. □

2 Asymptotic Properties of Local Polynomial Estimator

In this Section, we give detailed proofs of Lemma 2 and Theorem 1.

2.1 Conditions

The following conditions are sufficient for the asymptotic consistency of $\hat{g}_h(r)$.

[C1] There exists a C_λ such that the intensify function $0 \le \lambda(\mathbf{u}) \le C_\lambda$ for any $\mathbf{u} \in D_n$.

[C2] There exist positive constants c_g, C_g and C_f such that (a) $c_g \le g(r) \le C_g$; (b) $\max_{1\le j\le p+1} |f^{\{j\}}(r)| \le C_f$ for any $r \ge 0$ and that (c) $\int_0^\infty |g(s) - 1|\mathrm{d}s < C_g$.

[C3] It holds that (a) $\left|g^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k)\right| \le C_g$ for any $\mathbf{x}_j \in D_n$, $j = 1, \ldots, k$ and $k = 3, 4, 5, 6$; (b) $\int_{D_n} |g_0^{(3)}(\mathbf{x}, \mathbf{y}) - g(\|\mathbf{x}-\mathbf{y}\|)|\mathrm{d}\mathbf{x} \le C_g$; and (c) $\int_{D_n} |g_0^{(4)}(\mathbf{x}, \mathbf{y} + \mathbf{w}, \mathbf{w}) - g(\|\mathbf{x}\|)g(\|\mathbf{y}\|)|\mathrm{d}\mathbf{w} \le C_g$.

[C4] The kernel $K(x)$ has a bounded support, say $[-1, 1]$, such that $\int_{-1}^1 K(x)\mathrm{d}x = 1$.

[C5] As the bandwidth $h \to 0$ and $m|D_n|h(r + h)^{d-1} \to \infty$, there exists a constant $c_0 > 0$ such that

$$\eta_{\min}\left[\mathbf{Q}_{n,h}^{(k)}(r)\right](r + h)^{1-d} > c_0, \quad k = 1, 2,$$

where $\eta_{\min}(\mathbf{Q})$ denotes the smallest eigenvalue of the matrix $\mathbf{Q}$.

We need to make the following two additional assumptions for the asymptotic normality of $\hat{g}_h(r)$.

[N1] Either one of the following conditions are true (a) $m \to \infty$; or (b) the mixing coefficient satisfies $\alpha_X(s; h^{-1}, \infty) = O(s^{-d-\varepsilon})$ for some $\varepsilon > 0$.

[N2] There exists $\delta > 2d/\varepsilon$ such that $\left|g^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k)\right| \le C_g$ for any $\mathbf{x}_j \in D_n$, $j = 1, \ldots, k$, $k = 2, \ldots, 2(2 + \lceil\delta\rceil)$, where $\lceil\delta\rceil$ is the smallest integer greater than δ.

2.2 Sketch of the proof

Step 1 We first derive the asymptotic limit of solutions to $\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}) = \mathbf{0}$, namely, $\boldsymbol{\theta}^*$ defined in the (A.1) in the next subsection. As a result, Lemma A.1 gives the asymptotic bias of the local polynomial estimator by quantifying the distance between $\boldsymbol{\theta}^*$ and derivatives of $f(r) = \log[g(r)]$.

Step 2 Lemma A.2 gives the convergence rate of $\widehat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}^*$, which is of the order $O_P\left(\frac{1}{\sqrt{m|D_n|h}}\right)$ entry-wise;

Step 3 Establish the asymptotic normality of $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*$ through Lemmas A.3 to A.5.

Step 4 Finally use the delta method to derive asymptotic distribution of $\hat{g}_h(r) - g(r)$ based on distributional results of $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*$ given in Lemmas A.4-A.5, following the approach proposed in Biscio and Waagepetersen (2019) .

2.3 The asymptotic bias

Suppose there exists a vector $\boldsymbol{\theta}^* = (\theta_0^*, \theta_1^*, \ldots, \theta_p^*)^T \in \mathbb{R}^{p+1}$ such that

$$\int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v}) w_{r,h}(\|\mathbf{u}-\mathbf{v}\|) \left[g(\|\mathbf{u}-\mathbf{v}\|) - \tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|; \boldsymbol{\theta}^*)\right] \boldsymbol{G}_r(\|\mathbf{u}-\mathbf{v}\|) \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} = \mathbf{0}, \quad \text{(A.1)}$$

where $\tilde{g}_{r,h}(\cdot;\cdot)$ is defined in equation (3). Obviously $\boldsymbol{\theta}^*$ depends on n and h and r; i.e., $\boldsymbol{\theta}^* = \boldsymbol{\theta}^*_{n,h,r}$. Moreover, since $\boldsymbol{A}_h(t-r) = \mathbf{D}_h^{-1}\boldsymbol{G}_r(t)$, where $\mathbf{D}_h = \mathrm{diag}(1, h, \ldots, h^p)$, the above equation (A.1) is equivalent to

$$\int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v}) w_{r,h}(\|\mathbf{u}-\mathbf{v}\|) \left[g(\|\mathbf{u}-\mathbf{v}\|) - \tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|; \boldsymbol{\theta}^*)\right] \boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\| - r) \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} = \mathbf{0}.$$

The following Lemma quantifies the distance between $g(t)$ and $\tilde{g}_{r,h}(t; \boldsymbol{\theta}^*)$.

Lemma A.1. *Under conditions C1-C5, we have that as $h \to 0$,*

$$h^j \left[\theta_j^* - f^{\{j\}}(r)/j!\right] = O(h^{p+1}), \quad j = 0, 1, \ldots, p, \quad \text{(A.2)}$$

$$|g(t) - \tilde{g}_{r,h}(t; \boldsymbol{\theta}^*)| = O(h^{p+1}), \quad \text{for } t \in [r-h, r+h]. \quad \text{(A.3)}$$

Proof. Define function

$$g_{r,0}(t) = \exp\left\{f(r) + f^{\{1\}}(r)(t-r)/1! + \cdots + f^{\{p\}}(r)(t-r)^p/p!\right\}.$$

Then, from the Taylor's theorem with Lagrange's form of remainder

$$g(t) - g_{r,0}(t) = g(t)\left\{1 - \exp\left[\log g_{r,0}(t) - f(t)\right]\right\} = g(t)\left\{1 - \exp\left[-\frac{f^{\{p+1\}}(r_t^*)(t-r)^{p+1}}{(p+1)!}\right]\right\},$$

where r^* is between t and r, and it is straightforward to show that

$$\begin{aligned}
&\int_{D_n^2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\lambda(\mathbf{u})\lambda(\mathbf{v})\left[g(\|\mathbf{u}-\mathbf{v}\|) - g_{r,0}(\|\mathbf{u}-\mathbf{v}\|)\right]\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&= \int_{D_n^2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\lambda(\mathbf{u})\lambda(\mathbf{v})g(\|\mathbf{u}-\mathbf{v}\|) \\
&\qquad \left\{1 - \exp\left[-\frac{f^{\{p+1\}}(r^*_{\|\mathbf{u}-\mathbf{v}\|})(\|\mathbf{u}-\mathbf{v}\|-r)^{p+1}}{(p+1)!}\right]\right\}\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&\le C_\lambda^2 \int_{\mathbb{R}^d} K_h(\|\mathbf{s}\|-r)g(\|\mathbf{s}\|)\left|1 - \exp\left[-\frac{f^{\{p+1\}}(r^*_{\|\mathbf{s}\|})(\|\mathbf{s}\|-r)^{p+1}}{(p+1)!}\right]\right|\boldsymbol{A}_h(|\|\mathbf{s}\|-r|)\mathrm{d}\mathbf{s}
\end{aligned}$$

where the last inequality follows from condition C1 and $|D_n \cap (D_n - \mathbf{s})| \le |D_n|$ for any $\mathbf{s} \in \mathbb{R}^d$. Combining conditions C2(a)-(b), C4 and the fact that $|1-e^x| \le |x|e^{|x|}$, we have that as $h \to 0$,

$$\begin{aligned}
&(r+h)^{1-d}\int_{D_n^2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\lambda(\mathbf{u})\lambda(\mathbf{v})\left[g(\|\mathbf{u}-\mathbf{v}\|) - g_{r,0}(\|\mathbf{u}-\mathbf{v}\|)\right]\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&\le \frac{C_\lambda^2 C_g C_f}{(p+1)!}(r+h)^{1-d}\int_0^\infty |s-r|^{p+1}\exp\left\{\frac{C_f}{p+1}|s-r|^{p+1}\right\}K_h(s-r)\boldsymbol{A}_h(s-r)s^{d-1}\mathrm{d}s \\
&= O(h^{p+1})\int_{-r/h}^\infty |s|^{p+1}\exp\left\{\frac{C_f h^{p+1}}{p+1}|s|^{p+1}\right\}K(s)\boldsymbol{A}_1(s)\underbrace{\left(\frac{r+sh}{r+h}\right)^{d-1}}_{\le 1}\mathrm{d}s \\
&= O(h^{p+1}).
\end{aligned}$$

By the definition of $\boldsymbol{\theta}^*$ in (A.1), using the above equation, it is straightforward to see that

$$\begin{aligned}
(r+h)^{1-d}\int_{D_n^2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\lambda(\mathbf{u})\lambda(\mathbf{v})\left[\tilde{g}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*) - g_{r,0}(\|\mathbf{u}-\mathbf{v}\|)\right]\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
= O(h^{p+1}).
\end{aligned} \tag{A.4}$$

Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_p)^T$, define the continuously differentiable function $\boldsymbol{F} : \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ as follows

$$\boldsymbol{F}(\boldsymbol{a}) = (r+h)^{1-d} \int_{D_n^2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\lambda(\mathbf{u})\lambda(\mathbf{v}) q_{\boldsymbol{a},h}(\|\mathbf{u}-\mathbf{v}\|-r)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v},$$

where $q_{\boldsymbol{a},h}(t) = \exp\left\{\boldsymbol{a}^T \boldsymbol{A}_h(t-r)\right\} = \exp\left(a_0 + a_1(t-r)/h + \cdots + a_p(t-r)^p/h^p\right)$. Recall that $\tilde{g}_{r,h}(t;\boldsymbol{\theta}) = \exp\left[\theta_0 + \theta_1(t-r) + \cdots + \theta_p(t-r)^p\right] = \exp\left[(\mathbf{D}_h\boldsymbol{\theta})^T \boldsymbol{A}_h(t-r)\right]$, then equation (A.4) immediately yields that

$$\boldsymbol{F}(\mathbf{D}_h\boldsymbol{\theta}^*) - \boldsymbol{F}(f(r), hf^{\{1\}}(r)/1!, \ldots, h^p f^{\{p\}}(r)/p!) = O(h^{p+1}), \tag{A.5}$$

where $\mathbf{D}_h = \mathrm{diag}(1, h, \ldots, h^p)$. The Jacobian matrix of $\boldsymbol{F}(\boldsymbol{a})$ then becomes

$$\begin{aligned} \mathbf{J}(\boldsymbol{a}) &= \left[\frac{\partial \boldsymbol{F}(\boldsymbol{a})}{\partial a_0}, \ldots, \frac{\partial \boldsymbol{F}(\boldsymbol{a})}{\partial a_p}\right] \\ &= (r+h)^{1-d} \int_{D_n^2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\lambda(\mathbf{u})\lambda(\mathbf{v}) q_{\boldsymbol{a},h}(\|\mathbf{u}-\mathbf{v}\|-r)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}. \end{aligned}$$

Plugging in $\boldsymbol{a}_0 = (\log[g(r)], 0, \ldots, 0)^T$ back to the above equation, we have that $\mathbf{J}(\boldsymbol{a}_0) = g(r)\mathbf{Q}_{n,h}^{(1)}(r)$, where $\mathbf{Q}_{n,h}^{(1)}(r)$ is as defined in equation (11). Using conditions C2(a) and C5, we have that $\mathbf{J}(\boldsymbol{a})$ is strictly positive definite at $\boldsymbol{a}_0 = (\log[g(r)], 0, \ldots, 0)^T$ and hence $\det(\mathbf{J}(\boldsymbol{a}_0)) > 0$. Based on equation (A.5) and a simple application of the inverse function theorem (Burkill and Burkill, 2002, page 223) imply that $\boldsymbol{F}$ is invertible near $\boldsymbol{a}_0 = (f(r), 0, \ldots, 0)^T$ and as $h \to 0$, one has that,

$$h^j\left[\theta_j^* - f^{\{j\}}(r)/j!\right] = O(h^{p+1}), \quad j = 0, 1, \ldots, p.$$

Similar argument has been used in, e.g., Loader et al. (1996).

Finally, for any t satisfying $|t-r| \le h$, we have that

$$|g(t) - \tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)| = g(t)\left|1 - \exp\left[\theta_0^* + \theta_1^*(t-r) + \cdots + \theta_p^*(t-r)^p - f(t)\right]\right|$$

and

$$\theta_0^* + \theta_1^*(t-r) + \cdots + \theta_p^*(t-r)^p - f(t) = \sum_{j=0}^{p} \left[h^j(\theta_j^* - f^{\{j\}}(r)/j!)\right] \frac{(t-r)^j}{h^j} - f^{\{p+1\}}(r_t^*)\frac{|t-r|^{p+1}}{(p+1)!}$$
$$\leq \sum_{j=0}^{p} h^j(\theta_j^* - f^{\{j\}}(r)/j!) + \frac{C_f}{(p+1)!}h^{p+1}$$
$$= O(h^{p+1}).$$

Thus $|g(t) - \tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)| = O(h^{p+1})$, which concludes the proof. Note that, in particular,

$$|\exp(\theta_0^*) - g(r)| = g(r)|\exp(\theta_0^* - f(r)) - 1| \leq g(r)|\theta_0^* - f(r)|\exp(|\theta_0^* - f(r)|) = O(h^{p+1}). \quad \text{(A.6)}$$

□

2.4 Proof of Lemma 2

The proof of Lemma 2 follows immediately from the following Lemma A.2 and Lemma A.1 in the last subsection, because

$$|\hat{g}_h(r) - g(r)| \leq \exp(\theta_0^*)|\exp(\hat{\theta}_0 - \theta_0^*) - 1| + |\exp(\theta_0^*) - g(r)|.$$

So, in this section we just prove the following Lemma A.2.

Lemma A.2. *Under conditions C1-C5, we have that as $m|D_n|h(r+h)^{d-1} \to \infty$ and $h \to 0$,*

$$\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_h = O_P\left(\frac{1}{\sqrt{m|D_n|h(r+h)^{d-1}}}\right), \quad \text{(A.7)}$$

where the norm $\|\mathbf{x}\|_h^2 = x_0^2 + (hx_1)^2 + \cdots + (h^p x_p)^2$ for any $\mathbf{x} = (x_0, x_1, \ldots, x_p)^T \in \mathbb{R}^{p+1}$ and $\boldsymbol{\theta}^*$ is defined in equation (A.1).

Proof. It is straightforward to see that solving estimating equation (4) is equivalent to maximizing the composite log likelihood function

$$\tilde{L}_{r,h}(\boldsymbol{\theta}) = \frac{1}{m}\sum_{i=1}^{m}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\log\left[\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta})\right]$$
$$- \frac{1}{m(m-1)}\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}), \quad \text{(A.8)}$$

with respect to $\boldsymbol{\theta}$, because $\partial \tilde{L}_{r,h}(\boldsymbol{\theta})/\partial \boldsymbol{\theta} = \tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta})/m$. Note that the Hessian matrix

$$\tilde{\mathbf{H}}_{r,h}(\boldsymbol{\theta}) = \frac{\partial^2 \tilde{L}_{r,h}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} = -\sum\sum_{i \neq j = 1}^{m} \sum_{\mathbf{u} \in X_i} \sum_{\mathbf{v} \in X_j} \frac{w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)} \tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|; \boldsymbol{\theta}) \boldsymbol{G}_r(\|\mathbf{u}-\mathbf{v}\|) \boldsymbol{G}_r^T(\|\mathbf{u}-\mathbf{v}\|),$$

is negative definitive, which implies that $\tilde{L}_{r,h}(\boldsymbol{\theta})$ is a concave function of $\boldsymbol{\theta}$.

Let $J_{m,n}$ be a sequence of positive real numbers such that $J_{m,n} \to \infty$ as $m \to \infty$ and/or $n \to \infty$. We shall show that for any given $\varepsilon > 0$ there exists a large constant C_ϵ such that, for large m and/or large n,

$$\mathbb{P}\left\{ \sup_{\|\boldsymbol{\delta}\|_h = 1} \tilde{L}_{r,h}(\boldsymbol{\theta}^* + C_\epsilon J_{m,n}^{-1/2} \boldsymbol{\delta}) < \tilde{L}_{r,h}(\boldsymbol{\theta}^*) \right\} \geq 1 - \varepsilon. \tag{A.9}$$

Inequality (A.9) implies that with probability tending to 1 there is a local maximum, denoted as $\widehat{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}}_{m,n}$, in the the ellipsoid $\left\{ \boldsymbol{\theta}^* + C_\epsilon J_{m,n}^{-1/2} \boldsymbol{\delta} : \|\boldsymbol{\delta}\|_h = 1 \right\}$ centered at $\boldsymbol{\theta}^*$. It then follows that $J_{m,n} \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_h^2$ is bounded in probability; i.e., $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_h^2 = O_P(J_{m,n}^{-1})$.

To show (A.9), let $\boldsymbol{\delta}_{m,n}$ be any sequence in $\{\boldsymbol{\delta} \in \mathbb{R}^{p+1} : \|\boldsymbol{\delta}\|_h = 1\}$ and define a function of $z \geq 0$ as

$$H_{m,n}(z) = -\tilde{L}_{r,h}(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_{m,n}). \tag{A.10}$$

Then

$$H'_{m,n}(z) = -\frac{J_{m,n}^{-1/2}}{m} \boldsymbol{\delta}_{m,n}^T \tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_{m,n}), \tag{A.11}$$

$$H''_{m,n}(z) = -J_{m,n}^{-1} \boldsymbol{\delta}_{m,n}^T \tilde{\mathbf{H}}_{r,h}(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_{m,n}) \boldsymbol{\delta}_{m,n} \tag{A.12}$$

and by the Taylor's theorem

$$H_{m,n}(z) = H_{m,n}(0) + H'_{m,n}(0) z + H''_{m,n}(t_z) \frac{z^2}{2}$$

for any $z > 0$ and some $0 < t_z < z$, which implies that

$$\tilde{L}_{r,h}(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_{m,n}) - \tilde{L}_{r,h}(\boldsymbol{\theta}^*) = H_{m,n}(0) - H_{m,n}(z) = -z \left[H'_{m,n}(0) + \frac{z}{2} H''_{m,n}(t_z) \right]. \tag{A.13}$$

By definition, $H_{m,n}(z)$ is a convex function of z since $H''_{m,n}(z) \geq 0$ for any constant z. Therefore, to find a large enough C_ϵ so that (A.9) holds, it suffices to show that $H'_{m,n}(0) = O_p\left[H''_{m,n}(t_z)\right]$ for any $z > 0$. We first investigate $H'_{m,n}(0)$. By (A.11) and the definition of $\boldsymbol{\theta}^*$ in (A.1), we have that $\mathbb{E}\left[H'_{m,n}(0)\right] = -J_{m,n}^{-1/2}/m\boldsymbol{\delta}_{m,n}^T\mathbb{E}\left[\mathbf{U}_{r,h}(\boldsymbol{\theta}^*)\right] = 0$. Furthermore,

$$\operatorname{Var}\left[H'_{m,n}(0)\right] = \frac{J_{m,n}^{-1}}{m^2}\boldsymbol{\delta}_{m,n}^T\operatorname{Var}\left[\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}^*)\right]\boldsymbol{\delta}_{m,n} \tag{A.14}$$

and since $X_1, \ldots, X_m$ are independent replicates of the same Cox process

$$\begin{aligned}
\operatorname{Var}\left[\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}^*)\right] &= \operatorname{Var}\left[\sum_{i=1}^m \mathbf{Z}_{1,i} - \frac{1}{m-1}\sum\sum_{i\neq j=1}^m \mathbf{Z}_{2,i,j}(\boldsymbol{\theta}^*)\right] \\
&= m\operatorname{Var}(\mathbf{Z}_{1,1}) + \frac{2m}{(m-1)}\operatorname{Var}\left[\mathbf{Z}_{2,1,2}(\boldsymbol{\theta}^*)\right] \\
&+ \frac{4m(m-2)}{(m-1)}\operatorname{Cov}\left[\mathbf{Z}_{2,1,2}(\boldsymbol{\theta}^*), \mathbf{Z}_{2,1,3}(\boldsymbol{\theta}^*)\right] - 2m\operatorname{Cov}\left[\mathbf{Z}_{1,1}, \mathbf{Z}_{2,1,2}(\boldsymbol{\theta}^*)\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{Z}_{1,i} &= \sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} \boldsymbol{G}_r(\|\mathbf{u}-\mathbf{v}\|)w_{r,h}(\|\mathbf{u}-\mathbf{v}\|), \\
\mathbf{Z}_{2,i,j}(\boldsymbol{\theta}^*) &= \sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} \boldsymbol{G}_r(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta})w_{r,h}(\|\mathbf{u}-\mathbf{v}\|).
\end{aligned}$$

From a straightforward algebra and the definition of normalized joint intensities, we have that

$$\begin{aligned}
\operatorname{Var}(\mathbf{Z}_{1,1}) = &\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|) \\
&\times \boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\left[g^{(4)}(\mathbf{u}_1,\mathbf{v}_1,u_2,\mathbf{v}_2) - g(\|\mathbf{u}_1-\mathbf{v}_1\|)g(\|\mathbf{u}_2-\mathbf{v}_2\|)\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&+ 4\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|) \\
&\times \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&+ 2\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\left[w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)^T g(\|\mathbf{u}_1-\mathbf{v}_1\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1,
\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}\left[\mathbf{Z}_{1,1},\mathbf{Z}_{2,1,2}(\boldsymbol{\theta}^*)\right] &= \int_{D_n^4}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta})\\ &\quad\times\left[g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2)-g(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\ &+2\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{u}_1-\mathbf{v}_1\|)\\ &\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2,\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\left[\mathbf{Z}_{2,1,2}(\boldsymbol{\theta}^*)\right] &= \int_{D_n^4}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\\ &\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\left[g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1\right]\\ &\quad\times\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\ &+2\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\\ &\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)g(\|\mathbf{v}_1-\mathbf{u}_2\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\\ &+\int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}\left[\mathbf{Z}_{2,1,2}(\boldsymbol{\theta}^*),\mathbf{Z}_{2,1,3}(\boldsymbol{\theta}^*)\right] &= \int_{D_n^4}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\\ &\quad\times\left[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1\right]\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\\ &\quad\times\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2\\ &+\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\\ &\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2.\end{aligned}$$

Therefore, substituting the above terms in $\text{Var}\left[\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}^*)\right]$, we obtain

$$
\begin{aligned}
m^{-1}\text{Var}\left[\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}^*)\right] &= \int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\big[g^{(4)}(\mathbf{u}_1,\mathbf{v}_1,u_2,\mathbf{v}_2) \\
&\quad - g(\|\mathbf{u}_1-\mathbf{v}_1\|)g(\|\mathbf{u}_2-\mathbf{v}_2\|)\big]\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&- 4\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)[g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,u_2) - g(\|\mathbf{u}_1-\mathbf{v}_1\|)] \\
&\quad \times \tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&+ \frac{2}{m-1}\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*) \\
&\quad \times [g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1]\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&+ \frac{4(m-2)}{m-1}\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1] \\
&\times \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)^T d\mathbf{u}_1 d\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
&+ 4\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,u_2) \\
&\quad \times \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&- 4\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\left[2g(\|\mathbf{u}_1-\mathbf{v}_1\|) - \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\right] \\
&\quad \times \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&+ \frac{4}{m-1}\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*) \\
&\quad \times \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)[g(\|\mathbf{v}_1-\mathbf{u}_2\|)-1]\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&+ 2\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\left[w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 g(\|\mathbf{u}_1-\mathbf{v}_1\|)\ \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
&+ \frac{2}{m-1}\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1
\end{aligned}
$$

Note that by definition, $\boldsymbol{\delta}_{m,n}^T\boldsymbol{G}_r(t) = \boldsymbol{\eta}_{m,n}^T\boldsymbol{A}_h(t-r)$, where $\boldsymbol{\eta}_{m,n} = \mathbf{D}_h^{-1}\boldsymbol{\delta}_{m,n}$ and $\|\boldsymbol{\eta}_{m,n}\|^2 = \|\boldsymbol{\delta}_{m,n}\|_h = 1$, which implies that

$$
|\boldsymbol{\delta}_{m,n}^T\boldsymbol{G}_r(t)|^2 = \left|\boldsymbol{\eta}_{m,n}^T\boldsymbol{A}_h(t-r)\right|^2 \le \|\boldsymbol{A}_h(t-r)\|^2\|\boldsymbol{\eta}_{m,n}\|^2 = \|\boldsymbol{A}_h(t-r)\|^2 \le p+1
$$

for any $r-h \le t \le r+h$. Therefore, from (A.14) and under conditions C1, C2(a)-(b), C4 and equation (A.3), we obtain

$$
\begin{aligned}
mJ_{m,n}\mathrm{Var}\left[H'_{m,n}(0)\right] &= O(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)\Big|g^{(4)}(\mathbf{u}_1,\mathbf{v}_1,u_2,\mathbf{v}_2)\\
&\qquad - g(\|\mathbf{u}_1-\mathbf{v}_1\|)g(\|\mathbf{u}_2-\mathbf{v}_2\|)\Big|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&+O(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,u_2)-g(\|\mathbf{u}_1-\mathbf{v}_1\|)|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&+\frac{1}{m}O(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&+O(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1|d\mathbf{u}_1d\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2\\
&+O(1)\int_{D_n^3} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\\
&+\frac{1}{m}O(1)\int_{D_n^3} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)|g(\|\mathbf{v}_1-\mathbf{u}_2\|)-1|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\\
&+O(1)\int_{D_n^2}\left[w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\\
&= O(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)\Big|g^{(4)}(\mathbf{s},\mathbf{t}+\mathbf{w},\mathbf{w})-g(\|\mathbf{s}\|)g(\|\mathbf{t}\|)\Big|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&+O(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g^{(3)}(\mathbf{s},\mathbf{w})-g(\|\mathbf{s}\|)|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&+\frac{1}{m}O(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{w}\|)g(\|\mathbf{t}-\mathbf{s}+\mathbf{w}\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&+O(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{w}\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&+O(1)|D_n|^{-1}\int_{D_n^2} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\\
&+\frac{1}{m}O(1)|D_n|^{-1}\int_{D_n^2} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{s}-\mathbf{t}\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\\
&+O(1)|D_n|^{-1}\int_{D_n}\left[K_h(\|\mathbf{s}\|-r)\right]^2\mathrm{d}\mathbf{s}\\
&= O(1)|D_n|^{-1}\int_0^\infty K_h(s-r)s^{d-1}\mathrm{d}s\int_0^\infty K_h(t-r)t^{d-1}\mathrm{d}t\\
&+O(1)|D_n|^{-1}\int_0^\infty\left[K_h(s-r)\right]^2 s^{d-1}\mathrm{d}s,
\end{aligned}
$$

where the last equality follows from condition C3.

Finally, using condition C5, we have that

$$
\begin{aligned}
mJ_{m,n}\mathrm{Var}\left[H'_{m,n}(0)\right] &= O(1)|D_n|^{-1}\int_0^\infty K(s-r/h)s^{d-1}\mathrm{d}s\int_0^\infty K(t-r/h)t^{d-1}\mathrm{d}t \\
&+ O(1)|D_n|^{-1}h^{-1}\int_0^\infty \left[K(s-r/h)\right]^2 s^{d-1}\mathrm{d}s \\
&= O(1)|D_n|^{-1}\left[\int_{-r/h}^\infty K(s)(r+sh)^{d-1}\mathrm{d}s\right]^2 \\
&+ O(1)|D_n|^{-1}h^{-1}\int_{-r/h}^\infty \left[K(s)\right]^2(hs+r)^{d-1}\mathrm{d}s \\
&= O\left(\frac{(r+h)^{2d-2}}{|D_n|}\right) + O\left(\frac{(r+h)^{d-1}}{|D_n|h}\right) \\
&= O\left(\frac{(r+h)^{d-1}}{|D_n|h}\right).
\end{aligned}
$$

Combing with the fact that $\mathbb{E}\left[H'_{m,n}(0)\right]=0$, we have that

$$
H'_{m,n}(0) = O_P\left(\frac{(r+h)^{\frac{d-1}{2}}}{\sqrt{m|D_n|hJ_{m,n}}}\right). \tag{A.15}
$$

Now we proceed to study $H''_{m,n}(t_z)$. Let $\tilde{\boldsymbol{\theta}}^* = \tilde{\boldsymbol{\theta}}^*_{m,n} = \boldsymbol{\theta}^* + t_z J_{m,n}^{-1/2}\boldsymbol{\delta}_{m,n}$ and note that

$$
\mathrm{Var}\left[H''_{m,n}(t_z)\right] = \frac{J_{m,n}^{-2}}{m(m-1)}\mathrm{Var}\left[\sum_{\mathbf{u}\in X_1}\sum_{\mathbf{v}\in X_2} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\tilde{\boldsymbol{\theta}}^*)\left[\boldsymbol{\delta}_{m,n}^T G_r(\|\mathbf{u}-\mathbf{v}\|)\right]^2\right].
$$

Some tedious algebra gives that

$$
\begin{aligned}
& J_{m,n}^2 m(m-1)\mathrm{Var}\left[H''_{m,n}(t_z)\right] \\
& = 2\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2) w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|) w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|) \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) \tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\tilde{\boldsymbol{\theta}}^*) \\
& \qquad \times [g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1] \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
& \quad + 4\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2) w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|) w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|) \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) \\
& \qquad \times \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\tilde{\boldsymbol{\theta}}^*) g(\|\mathbf{v}_1-\mathbf{u}_2\|) \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
& \quad + 2\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1) w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|) \tilde{g}_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) g(\|\mathbf{u}_1-\mathbf{v}_1\|) \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^4 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
& \quad + 4(m-2)\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2) w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|) w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1] \\
& \qquad \times \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) \tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\tilde{\boldsymbol{\theta}}^*) \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_2-\mathbf{v}_2\|)\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
& \quad + 4(m-2)\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2) w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|) w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|) \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) \\
& \qquad \times \tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\tilde{\boldsymbol{\theta}}^*) \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 \left[\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{u}_2\|)\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2.
\end{aligned}
$$

Recall that we have shown that $|\boldsymbol{\delta}_{m,n}^T \boldsymbol{G}_r(t)|^2 \leq p+1$ for any $r-h \leq t \leq r+h$. Then under conditions C1, C2(a)-(b), C3 and equation (A.3), we can further simplify $J_{m,n}^2 m(m-$

1)Var $\left[H''_{m,n}(t_z)\right]$ as follows

$$
\begin{aligned}
&J_{m,n}^2 m(m-1)\text{Var}\left[H''_{m,n}(t_z)\right] \\
&\quad = O(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&\qquad + O(1)\int_{D_n^3} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 + O(1)\int_{D_n^2} w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
&\qquad + mO(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1|d\mathbf{u}_1 d\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
&\qquad + mO(1)\int_{D_n^3} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&\quad = O(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{w}\|)g(\|\mathbf{t}-\mathbf{s}+\mathbf{w}\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w} \\
&\qquad + O(1)|D_n|^{-1}\int_{D_n^2} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} + O(1)|D_n|^{-1}\int_{D_n}\left[K_h(\|\mathbf{s}\|-r)\right]^2\mathrm{d}\mathbf{s} \\
&\qquad + mO(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{w}\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w} \\
&\qquad + mO(1)|D_n|^{-1}\int_{D_n^2} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
&\quad = O(1)|D_n|^{-1}\int_0^\infty \left[K_h(s-r)\right]^2 s^{d-1}\mathrm{d}s + mO(1)|D_n|^{-1}\left[\int_0^\infty K_h(s-r)s^{d-1}\mathrm{d}s\right]^2 \\
&\quad = O(1)|D_n|^{-1}h^{-1}\int_{-r/h}^\infty \left[K(s)\right]^2 (r+sh)^{d-1}\mathrm{d}s + mO(1)|D_n|^{-1}\left[\int_{-r/h}^\infty K(s)(r+sh)^{d-1}\mathrm{d}s\right]^2
\end{aligned}
$$

Then, by condition C4, we finally have that

$$
J_{m,n}^2 m(m-1)\text{Var}\left[H''_{m,n}(t_z)\right] = \frac{(r+h)^{d-1}}{|D_n|h}O(1) + \frac{m(r+h)^{2d-2}}{|D_n|}O(1),
$$

which implies that

$$
\text{Var}\left[H''_{m,n}(t_z)\right] = O\left(\frac{(r+h)^{d-1}}{J_{m,n}^2 m^2|D_n|h} + \frac{(r+h)^{2d-2}}{J_{m,n}^2 m|D_n|}\right). \tag{A.16}
$$

On the other hand, we have that

$$
\mathbb{E}\left[H''_{m,n}(t_z)\right] = J_{m,n}^{-1}\int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)\left[\boldsymbol{\delta}_{m,n}^T\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1.
$$

Observe that by definition $\boldsymbol{\delta}_{m,n}^T\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|) = \boldsymbol{\eta}_{m,n}^T\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1| - r)$ with $\|\boldsymbol{\eta}_{m,n}\|^2 = 1$

and the fact that the smallest eigenvalue satisfies the condition

$$\eta_{\min}\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]=\inf_{\|\boldsymbol{\eta}\|^2=1}\boldsymbol{\eta}^T\mathbf{Q}_{n,h}^{(1)}(r)\boldsymbol{\eta}.$$

Using definition of $\mathbf{Q}_{n,h}^{(1)}(r)$ in condition C5, we have that

$$\begin{aligned}\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]-g(r)\eta_{\min}\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]&\geq\int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\\&\quad\times\left[\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)-g(r)\right]\left[\boldsymbol{\delta}_{m,n}^T\boldsymbol{G}_r(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\\&=O(1)\int_{D_n^2}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\left|\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)-g(r)\right|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\\&=O(1)\int_0^\infty K_h(s-r)\left|\tilde{g}_{r,h}(s;\tilde{\boldsymbol{\theta}}^*)-g(r)\right|s^{d-1}\mathrm{d}s.\end{aligned}$$

Note that by the definition of $\tilde{\boldsymbol{\theta}}^*=\boldsymbol{\theta}^*+t_zJ_{m,n}^{-1/2}\boldsymbol{\delta}_{m,n}$ and equation (A.2) of Lemma A.1, it is straightforward to show that

$$\left|\tilde{g}_{r,h}(t;\tilde{\boldsymbol{\theta}}^*)-g(r)\right|=g(r)\left|\exp\left(\sum_{j=0}^p\left(\tilde{\theta}_j^*-f^{\{j\}}(r)/j!\right)(t-r)^j-f^{(p+1)}(r_t^*)\frac{(t-r)^{p+1}}{(p+1)!}\right)-1\right|$$

and for any $r-h\leq t\leq r+h$,

$$\begin{aligned}\left(\tilde{\theta}_j^*-f^{\{j\}}(r)/j!\right)(t-r)^j&=h^j\left(\theta_j^*-f^{\{j\}}(r)/j!\right)\frac{(t-r)^j}{h^j}+z_0J_{m,n}^{-1/2}h^j\delta_{m,n,j+1}\frac{(t-r)^j}{h^j}\\&\leq h^j\left(\theta_j^*-f^{\{j\}}(r)/j!\right)\frac{(t-r)^j}{h^j}+z_0J_{m,n}^{-1/2}\|\boldsymbol{\delta}_{m,n}\|_h\\&=O(h^{p+1}+J_{m,n}^{-1/2}),\end{aligned}$$

which implies that

$$\left|\tilde{g}_{r,h}(t;\tilde{\boldsymbol{\theta}}^*)-g(r)\right|=g(r)O\left(h^{p+1}+J_{m,n}^{-1/2}\right).$$

Therefore, we have that, under conditions C4-C5,

$$\begin{aligned}(r+h)^{1-d}\left\{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]-g(r)\eta_{\min}\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]\right\}&\geq O(J_{m,n}^{-1/2}+h^{p+1})\int_{-r/h}^\infty K(s)\underbrace{\left(\frac{r+sh}{r+h}\right)^{d-1}}_{\leq 1}\mathrm{d}s\\&=O(J_{m,n}^{-1/2}+h^{p+1}).\end{aligned}$$

By condition C2(a) and C5, the above equation gives that

$$(r+h)^{1-d}\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right] \geq c_g c_0 + O(J_{m,n}^{-1/2} + h^{p+1}). \tag{A.17}$$

Hence for the constant $c = c_0 c_g$, by an application of the Chebyshev's inequality we have that

$$\begin{aligned}
&\mathbb{P}\left\{(r+h)^{1-d}J_{m,n}H''_{m,n}(t_z) < \frac{c}{2}\right\} \\
&= \mathbb{P}\left\{\frac{J_{m,n}H''_{m,n}(t_z)}{(r+h)^{d-1}} - \frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}} < \frac{c}{2} - \frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}}\right\} \\
&\leq \mathbb{P}\left\{\frac{\left|J_{m,n}H''_{m,n}(t_z) - \mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]\right|}{(r+h)^{d-1}} > \left|\frac{c}{2} - \frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}}\right|\right\} I\left\{\frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}} > \frac{c}{2}\right\} \\
&\quad + I\left\{\frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}} \leq \frac{c}{2}\right\} \\
&\leq \frac{\mathrm{Var}\left[(r+h)^{1-d}J_{m,n}H''_{m,n}(t_z)\right]}{\left|c/2 - \mathbb{E}\left[(r+h)^{1-d}J_{m,n}H''_{m,n}(t_z)\right]\right|^2} I\left\{\frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}} > \frac{c}{2}\right\} + I\left\{\frac{\mathbb{E}\left[J_{m,n}H''_{m,n}(t_z)\right]}{(r+h)^{d-1}} \leq \frac{c}{2}\right\} \\
&= O(1)\mathrm{Var}\left[(r+h)^{1-d}J_{m,n}H''_{m,n}(t_z)\right] + o(1) \\
&= O\left(\frac{(r+h)^{1-d}}{m^2|D_n|h} + \frac{1}{m|D_n|}\right) + o(1),
\end{aligned}$$

where the last equality follows from equations (A.16) and (A.17) as $J_{m,n} \to \infty$ and $h \to 0$. Therefore, as long as $m|D_n|h(r+h)^{d-1} \to \infty$, $J_{m,n} \to \infty$ and $h \to 0$, we have that

$$\mathbb{P}\left\{(r+h)^{1-d}J_{m,n}H''_{m,n}(t_z) \geq \frac{c_0 c_g}{2}\right\} \to 1, \tag{A.18}$$

where c_g and c_0 are constants defined in conditions C2(a) and C6, respectively.

We have already shown in equation (A.15) that

$$H'_{m,n}(0) = O_P\left(\frac{(r+h)^{(d-1)/2}}{\sqrt{J_{m,n}m|D_n|h}}\right),$$

hence as long as $\frac{J_{m,n}(r+h)^{d-1}}{m|D_n|h} = O(1)$, we have that $H'_{m,n}(0) = O_P(H''_{m,n}(t_z))$. In other words, by taking $J_{m,n} = m|D_n|h(r+h)^{1-d}$, we have that

$$\mathbb{P}\left\{|H'_{m,n}(0)| \geq \frac{z}{2}H''_{m,n}(t_z)\right\} < \epsilon,$$

where ϵ can be arbitrary small by choosing z and m and/or n large enough. Therefore, with $J_{m,n} = m|D_n|h(r+h)^{1-d}$, for any given $\epsilon > 0$, there exists $z_\epsilon > 0$ such that for large m and/or n,

$$\mathbb{P}\left\{\tilde{L}_{r,h}(\boldsymbol{\theta}^* + z_\epsilon J_{m,n}^{-1/2}\boldsymbol{\delta}_{m,n}) < \tilde{L}_{r,h}(\boldsymbol{\theta}^*)\right\} = \mathbb{P}\left\{z_\epsilon H'_{m,n}(0) + \frac{z_\epsilon^2}{2}H''_{m,n}(t_{z_\epsilon}) > 0\right\} \geq 1 - \epsilon.$$

Thus, (A.9) holds, which completes the proof of equation (A.7). □

2.5 Proof of Theorem 1

Define two random vectors

$$\mathbf{Z}_1 = \frac{1}{m}\sum_{i=1}^{m}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r), \tag{A.19}$$

$$\mathbf{Z}_2(\boldsymbol{\theta}^*) = \frac{1}{m(m-1)}\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r), \tag{A.20}$$

By definition of $\boldsymbol{\theta}^*$ in (A.1), we have that

$$\mathbb{E}\mathbf{Z}_1 = \mathbb{E}\mathbf{Z}_2 = \int_{D_n^2}\lambda(\mathbf{u})\lambda(\mathbf{v})w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}. \tag{A.21}$$

Lemma A.3. *Under conditions C1-C5, as $h \to 0$ and $m|D_n|h(r+h)^{d-1} \to \infty$, we have that,*

$$\begin{aligned}
(m|D_n|h)\mathrm{Var}\,(\mathbf{Z}_1) &= 2g(r)\mathbf{Q}_{n,h}^{(2)}(r) + O(h(r+h)^{d-1}), &\text{(A.22)}\\
(m|D_n|h)\mathrm{Var}\,[\mathbf{Z}_2(\boldsymbol{\theta}^*)] &= \frac{2}{m-1}g^2(r)\mathbf{Q}_{n,h}^{(2)}(r) + O(h(r+h)^{d-1}), &\text{(A.23)}\\
(m|D_n|h)\mathrm{Cov}\,[\mathbf{Z}_1,\mathbf{Z}_2(\boldsymbol{\theta}^*)] &= O(h(r+h)^{d-1}), &\text{(A.24)}
\end{aligned}$$

where $\mathbf{Q}_{n,h}^{(2)}(r)$ is as defined in equation (11) and the convergence is entry-wise.

Proof. Under conditions C1-C5, using similar arguments as those in the proof of Lemma A.2, we can immediately show that

$$\frac{m\mathrm{Var}\left(\mathbf{Z}_1\right)}{(r+h)^{d-1}} = (r+h)^{1-d}\int_{D_n^4}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)[g^{(4)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2,\mathbf{v}_2)$$

$$- g(\|\mathbf{u}_1-\mathbf{v}_1\|)g(\|\mathbf{u}_2-\mathbf{v}_2\|)]\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_2-\mathbf{v}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2$$

$$+ 4(r+h)^{1-d}\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,u_2)$$

$$\times \boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{u}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2$$

$$+ 2\int_{D_n^2}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)}{(r+h)^{d-1}}\left[w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 g(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1$$

$$= 2\int_{D_n^2}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)}{(r+h)^{d-1}}\left[w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 g(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1$$

$$+ O(|D_n|^{-1})$$

$$= 2g(r)\frac{\{1+O(h)\}}{(r+h)^{d-1}}\int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\left[w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1$$

$$+ O(|D_n|^{-1}),$$

where the last equality follows from C4 and the fact that $|g(t)-g(r)| = O(h)$ for any $r-h \leq t \leq r+h$ as $h \to 0$. Similarly, we can show that under conditions C1-C5, we have that

$$\begin{aligned}
\frac{m\text{Var}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]}{(r+h)^{d-1}} &= \frac{2(r+h)^{1-d}}{m-1}\int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)g(\|\mathbf{u}_1-\mathbf{v}_1\|)\\
&\quad\times \boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\\
&\quad+\frac{4(r+h)^{1-d}}{m-1}\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\\
&\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)g(\|\mathbf{v}_1-\mathbf{u}_2\|)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{u}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\\
&\quad+\frac{4(m-2)}{m-1}\int_{D_n^3}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)}{(r+h)^{d-1}}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)\\
&\quad\times \boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{u}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\\
&\quad+\frac{2}{m-1}\int_{D_n^4}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)}{(r+h)^{d-1}}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1]\\
&\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_2-\mathbf{v}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&\quad+\frac{4(m-2)}{m-1}\int_{D_n^4}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)}{(r+h)^{d-1}}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1]\\
&\quad\times\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_2-\mathbf{v}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2\\
&=\int_{D_n^2}\frac{2\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)}{m-1}\frac{w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)}{(r+h)^{d-1}}\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1+O(|D_n|^{-1})\\
&=\frac{2g^2(r)\left\{1+O(h)\right\}}{(r+h)^{d-1}}\int_{D_n^2}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)}{m-1}w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1+O(|D_n|^{-1}),
\end{aligned}$$

where the last equality holds because (A.2) implies that for any $r-h\leq t\leq r+h$,

$$|\tilde{g}_{t,h}^2(t;\boldsymbol{\theta}^*)-g^2(r)|=g^2(r)\left|\exp(2\left\{\log\tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)-f(r)\right\})-1\right|=O(h^{p+1}),$$

and that

$$\frac{m\mathrm{Cov}\left[\mathbf{Z}_1,\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]}{(r+h)^{d-1}} = 2\times 2\int_{D_n^3}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)}{(r+h)^{d-1}}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)$$
$$\times \boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{u}_2\|-r)^T\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2$$
$$+2\int_{D_n^4}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)}{(r+h)^{d-1}}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)$$
$$\times \boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_2-\mathbf{v}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2$$
$$-2\int_{D_n^4}\frac{\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)}{(r+h)^{d-1}}w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)g(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)$$
$$\times \boldsymbol{A}_h(\|\mathbf{u}_1-\mathbf{v}_1\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}_2-\mathbf{v}_2\|-r)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2$$
$$=O(|D_n|^{-1}).$$

Combining above three equalities, we can conclude that as $h\to 0$

$$\begin{aligned}
(m|D_n|h)\mathrm{Var}\left(\mathbf{Z}_1\right) &= 2g(r)\mathbf{Q}_{n,h}^{(2)}(r)+O(h(r+h)^{d-1}),\\
(m|D_n|h)\mathrm{Var}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*)\right] &= \frac{2}{m-1}g^2(r)\mathbf{Q}_{n,h}^{(2)}(r)+O(h(r+h)^{d-1}),\\
(m|D_n|h)\mathrm{Cov}\left[\mathbf{Z}_1,\mathbf{Z}_2(\boldsymbol{\theta}^*)\right] &= O(h(r+h)^{d-1}),
\end{aligned}$$

where $\mathbf{Q}_{n,h}^{(2)}(r)$ is as defined in equation (11) and the convergence is entry-wise. □

Lemma A.4. *Under conditions C1-C5 and N1-N2, we have that, as $h\to 0$ and $m|D_n|h(r+h)^{d-1}\to\infty$,*

$$\sqrt{m|D_n|h}\boldsymbol{\Sigma}_Z^{-1/2}(\boldsymbol{\theta}^*)\left[\mathbf{Z}_1-\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]\xrightarrow{\mathcal{D}}N\left(\mathbf{0},\mathbf{I}\right), \tag{A.25}$$

where $\boldsymbol{\Sigma}_Z(\boldsymbol{\theta}^*)=2(m-1+g(r))/(m-1)g(r)\mathbf{Q}_{n,h}^{(2)}(r)$ with $\mathbf{Q}_{n,h}^{(2)}(r)$ defined in equation (11).

Proof. By equation (A.24) of Lemma A.3, we can see that $\mathbf{Z}_1$ and $\mathbf{Z}_2(\boldsymbol{\theta}^*)$ are asymptotically uncorrelated as $h\to 0$. Hence, it suffices to consider asymptotic normality of $(r+h)^{(1-d)/2}\mathbf{Z}_1$ and $(r+h)^{(1-d)/2}\mathbf{Z}_2(\boldsymbol{\theta}^*)$ separately. We divide our discussions into two case scenarios: (1) $m\to\infty$ and (2) m is fixed.

Case I: when $m \to \infty$. In this case, from equations (A.22)-(A.23), we can see that as $m \to \infty$ and $h \to 0$,

$$\sqrt{m|D_n|h}\,\{\mathbf{Z}_2(\boldsymbol{\theta}^*) - \mathbb{E}\,[\mathbf{Z}_2(\boldsymbol{\theta}^*)]\} = o_P((r+h)^{(d-1)/2}),$$

which implies that

$$\sqrt{m|D_n|h}\,[\mathbf{Z}_1 - \mathbf{Z}_2(\boldsymbol{\theta}^*)] = \sqrt{m|D_n|h}\,(\mathbf{Z}_1 - \mathbb{E}\mathbf{Z}_1) + o_p((r+h)^{(d-1)/2}),$$

since $\mathbb{E}\mathbf{Z}_1 = \mathbb{E}\,[\mathbf{Z}_2(\boldsymbol{\theta}^*)]$. Let $\mathbf{Y}_i = (r+h)^{(1-d)/2}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)$, then $(r+h)^{(1-d)/2}\mathbf{Z}_1 = \frac{1}{m}\sum_{i=1}^m \mathbf{Y}_i$. By definition, $\mathbf{Y}_i$'s are independent and identically distributed, thus it immediately follows from the standard multivariate central limit theorem that as $h \to 0$ and $m \to \infty$,

$$[\mathrm{Var}(\mathbf{Z}_1)]^{-1/2}\,(\mathbf{Z}_1 - \mathbb{E}\mathbf{Z}_1) \xrightarrow{\mathcal{D}} N(\mathbf{0}, \mathbf{I}),$$

which coincides with (A.25) after plugging (A.22) back to the above equation and use (A.23)-(A.24) to obtain the asymptotic variance of $\sqrt{m|D_n|h}\,[\mathbf{Z}_1 - \mathbf{Z}_2(\boldsymbol{\theta}^*)]$.

Case II: when m is fixed. In this case, condition $m|D_n|h \to \infty$ requires that $|D_n| \to \infty$. In other words, we need to consider the case where the observation window of the point processes is expanding. Define a partition of $\mathbb{R}^d = \cup_{\mathbf{t}\in\mathbb{Z}^d}\Delta_h(\mathbf{t})$, where $\Delta_h(\mathbf{t}) = \prod_{k=1}^d (h^{-1/d}(r+h)^{1/d-1}(t_k - 1/2), h^{-1/d}(r+h)^{1/d-1}(t_k + 1/2)]$. Note that by this definition, $\Delta_h(\mathbf{t}_1) \cap \Delta_h(\mathbf{t}_2) = \emptyset$ if $\mathbf{t}_1 \neq \mathbf{t}_2 \in \mathbb{Z}^d$. Define random vectors

$$\mathbf{Y}_{1,n}(\mathbf{t}) = \frac{|D_n|h}{m}\sum_{i=1}^m \sum_{\mathbf{u}\in X_i\cap\Delta_h(\mathbf{t}),\mathbf{v}\in X_i}\sum^{\neq} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r),$$

$$\mathbf{Y}_{2,n}(\mathbf{t}) = \frac{|D_n|h}{m(m-1)}\sum\sum_{i\neq j}\sum_{\mathbf{u}\in X_i\cap\Delta_h(\mathbf{t}),\mathbf{v}\in X_j}\sum w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r).$$

Then by definition, we have that

$$\mathbf{Z}_1 = \frac{1}{|D_n|h}\sum_{\mathbf{t}\in\mathcal{T}_n}\mathbf{Y}_{1,n}(\mathbf{t}), \quad \mathbf{Z}_2(\boldsymbol{\theta}^*) = \frac{1}{|D_n|h}\sum_{\mathbf{t}\in\mathcal{T}_n}\mathbf{Y}_{2,n}(\mathbf{t}),$$

where $\mathcal{T}_n = \{\mathbf{t} \in \mathbb{Z}^d : \Delta_h(\mathbf{t}) \cap D_n \neq \emptyset\}$.

Under conditions C4-C5, it is straightforward to see that there exists a constant C_1 such that

$$|D_n| h w_{r,h}(\|\mathbf{u}-\mathbf{v}\|) |\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\| - r)| \leq C_1 I(|\|\mathbf{u}-\mathbf{v}\| - r| < h).$$

A simple application of the Jensen's inequality gives that $(m^{-1}\sum_{i=1}^m |x_i|)^{2+\lceil\delta\rceil} \leq m^{-1}\sum_{i=1}^m |x_i|^{2+\lceil\delta\rceil}$ (note that $f(x) = x^{2+\lceil\delta\rceil}$ is convex for $x > 0$)

$$\begin{aligned}
\mathbb{E}\,|\mathbf{Y}_{1,n}(t)|^{2+\lceil\delta\rceil} &\leq \frac{1}{m}\sum_{i=1}^m \mathbb{E}\left|\sum_{\mathbf{u}\in X_i\cap\Delta_h(t),\mathbf{v}\in X_i}^{\neq}\sum |D_n| h w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\| - r)\right|^{2+\lceil\delta\rceil} \\
&= \mathbb{E}\left|\sum_{\mathbf{u}\in X_1\cap\Delta_h(t),\mathbf{v}\in X_1}^{\neq}\sum hK_h(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\| - r)\right|^{2+\lceil\delta\rceil} \\
&\leq \mathbb{E}\left\{\sum_{\mathbf{u}\in X_1\cap\Delta_h(t),\mathbf{v}\in X_1}^{\neq}\sum hK_h(\|\mathbf{u}-\mathbf{v}\|)|\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\| - r)|\right\}^{2+\lceil\delta\rceil} \\
&\leq C_1^{2+\lceil\delta\rceil}\mathbb{E}\left\{\sum_{\mathbf{u}\in X_1\cap\Delta_h(t),\mathbf{v}\in X_1}^{\neq}\sum I\left(|\|\mathbf{u}-\mathbf{v}\| - r| < h\right)\right\}^{2+\lceil\delta\rceil},
\end{aligned}$$

where the last expectation is essentially bounded by sums of integrals involving $\lambda(\mathbf{u})$, $g(s)$, $g^{(k)}(\mathbf{u}_1,\ldots,\mathbf{u}_k)$, $k = 3,\ldots,2(2+\lceil\delta\rceil)$. Specifically, note that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\mathbf{u}\in X_1\cap\Delta_h(t),\mathbf{v}\in X_1}^{\neq}\sum I\left(|\|\mathbf{u}-\mathbf{v}\| - r| < h\right)\right] \\
&= \int_{\Delta_h(t)}\int_{D_n} \lambda(\mathbf{u})\lambda(\mathbf{v})g_0(\|\mathbf{u}-\mathbf{v}\|) I\left(|\|\mathbf{u}-\mathbf{v}\| - r| < h\right) \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^d} g_0(\|\mathbf{h}\|) I\left(|\|\mathbf{h}\| - r| < h\right)\left[\int_{\Delta_h(t)} \lambda(\mathbf{u})\lambda(\mathbf{u}-\mathbf{h}) I\left(\mathbf{u}-\mathbf{h} \in D_n\right)\mathrm{d}\mathbf{u}\right]\mathrm{d}\mathbf{h} \\
&= O(1)\int_{\mathbb{R}^d} g_0(\|\mathbf{h}\|) I\left(|\|\mathbf{h}\| - r| < h\right)\left[\int_{\Delta_h(t)} I\left(\mathbf{u}-\mathbf{h} \in D_n\right)\mathrm{d}\mathbf{u}\right]\mathrm{d}\mathbf{h} \\
&= O(1)|\Delta_h(t)|\int_{\mathbb{R}^d} g_0(\|\mathbf{h}\|) I\left(|\|\mathbf{h}\| - r| < h\right)\mathrm{d}\mathbf{h} \\
&= O(1)h^{-1}(r+h)^{1-d}\int_0^\infty I\left(|s-r| < h\right) s^{d-1}\mathrm{d}s = O(1).
\end{aligned}$$

All other terms can be similarly shown to be of the same order under conditions C1-C3 and condition N2, hence these integrals bounded for any $d \geq 1$ and uniformly in t and n. Recall that δ is defined in condition N2. Therefore, we have that

$$\sup_{n\geq 1}\sup_{\mathbf{t}\in\mathcal{T}_n} \mathbb{E}\left|\mathbf{Y}_{1,n}(\mathbf{t})\right|^{2+\delta} \leq \left(\sup_{n\geq 1}\sup_{\mathbf{t}\in\mathcal{T}_n} \mathbb{E}\left|\mathbf{Y}_{1,n}(\mathbf{t})\right|^{2+\lceil\delta\rceil}\right)^{\frac{2+\delta}{2+\lceil\delta\rceil}} < \infty. \tag{A.26}$$

Similarly, using equation (A.3) in Lemma A.1 and condition C2(a), we have that $\tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)$ is also uniformly bounded and following similar arguments as above, we can show that

$$\sup_{n\geq 1}\sup_{\mathbf{t}\in\mathcal{T}_n} \mathbb{E}\left|\mathbf{Y}_{2,n}(\mathbf{t})\right|^{2+\delta} \leq \left(\sup_{n\geq 1}\sup_{\mathbf{t}\in\mathcal{T}_n} \mathbb{E}\left|\mathbf{Y}_{2,n}(\mathbf{t})\right|^{2+\lceil\delta\rceil}\right)^{\frac{2+\delta}{2+\lceil\delta\rceil}} < \infty. \tag{A.27}$$

Note that the total number of disjoint partitions $\Delta_h(\mathbf{t}) \cap D_n \neq \emptyset$ is of the order $|D_n|h(r+h)^{d-1}$, hence we can check that, using equations (A.22)-(A.23),

$$\frac{\mathrm{Var}\left[\sum_{t\in\mathcal{T}_n}\mathbf{Y}_{1,n}(t)\right]}{|D_n|h(r+h)^{d-1}} = \frac{\mathrm{Var}\left(|D_n|h\ \mathbf{Z}_1\right)}{|D_n|h(r+h)^{d-1}} = |D_n|h(r+h)^{1-d}\mathrm{Var}\left(\mathbf{Z}_1\right),$$

$$\frac{\mathrm{Var}\left[\sum_{t\in\mathcal{T}_n}\mathbf{Y}_{2,n}(t)\right]}{|D_n|h(r+h)^{d-1}} = \frac{\mathrm{Var}\left[|D_n|h\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]}{|D_n|h(r+h)^{d-1}} = |D_n|h(r+h)^{1-d}\mathrm{Var}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*)\right],$$

both of above matrices have strictly positive eigenvalues under condition C5 and Lemma A.3. Therefore, using conditions $N1(b)$ and $N2$, together with inequalities (A.26)-(A.27), it follows from Theorem 1 of Biscio and Waagepetersen (2019) that as $|D_n|h(r+h)^{d-1} \to \infty$,

$$\left\{\mathrm{Var}\left[\sum_{t\in\mathcal{T}_n}\mathbf{Y}_{k,n}(t)\right]\right\}^{-1/2}\sum_{t\in\mathcal{T}_n}\left[\mathbf{Y}_{k,n}(t) - \mathbb{E}\mathbf{Y}_{k,n}(t)\right] \xrightarrow{\mathcal{D}} N(\mathbf{0},\mathbf{I}), \quad k=1,2,$$

which is equivalent to stating that

$$\left[\mathrm{Var}(\mathbf{Z}_1)\right]^{-1/2}\left(\mathbf{Z}_1 - \mathbb{E}\mathbf{Z}_1\right) \xrightarrow{\mathcal{D}} N(\mathbf{0},\mathbf{I}),$$

and

$$\left\{\mathrm{Var}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]\right\}^{-1/2}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*) - \mathbb{E}\mathbf{Z}_2(\boldsymbol{\theta}^*)\right] \xrightarrow{\mathcal{D}} N(\mathbf{0},\mathbf{I}).$$

Recall that by Lemma A.3, $\mathbf{Z}_1$ and $\mathbf{Z}_2(\boldsymbol{\theta}^*)$ have finite variances and are asymptotically independent as $h \to 0$, and that $\mathbb{E}\mathbf{Z}_1 = \mathbb{E}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]$ by definition, we can conclude that

$$\left\{\mathrm{Var}(\mathbf{Z}_1) + \mathrm{Var}\left[\mathbf{Z}_2(\boldsymbol{\theta}^*)\right]\right\}^{-1/2}\left[\mathbf{Z}_1 - \mathbf{Z}_2(\boldsymbol{\theta}^*)\right] \xrightarrow{\mathcal{D}} N(\mathbf{0},\mathbf{I}),$$

which coincides with (A.25) after plugging (A.22)-(A.23) back to the above equation. The proof is complete. □

Lemma A.5. *Denote $\widehat{\boldsymbol{\theta}}$ as the solution to estimating equations (6), then under conditions C1-C5, N1-N2, we have that, as $h \to 0$ and $m|D_n|h(r+h)^{d-1} \to \infty$,*

$$\mathbf{D}_h(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) \equiv \begin{bmatrix} (\hat{\theta}_0 - \theta_0^*) \\ h(\hat{\theta}_1 - \theta_1^*) \\ \vdots \\ h^p(\hat{\theta}_p - \theta_p^*) \end{bmatrix} = \left[g(r)\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\left[\mathbf{Z}_1 - \mathbf{Z}_2(\boldsymbol{\theta}^*) + o_p\left(\sqrt{\frac{(r+h)^{1-d}}{m|D_n|h}}\right)\right] \quad (A.28)$$

where $\mathbf{Z}_1$ and $\mathbf{Z}_2(\boldsymbol{\theta}^*)$ are defined in (A.19) *and* (A.20)*, respectively.*

Proof. By the definition of $\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta})$ in (6), since $\boldsymbol{G}_r(t) = \mathbf{D}_h \boldsymbol{A}_h(t-r)$, solving $\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta}) = \mathbf{0}$ is equivalent to solving $\tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}) = \mathbf{0}$ for $\boldsymbol{\theta}$, where $\tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}) = \mathbf{D}_h\tilde{\mathbf{U}}_{r,h}(\boldsymbol{\theta})$; i.e.,

$$\begin{aligned}\tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}) = &\frac{1}{m}\sum_{i=1}^{m}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r) \\ &- \frac{1}{m(m-1)}\sum\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}).\end{aligned}$$

Using the first order Taylor expansion, we can show that

$$\underbrace{\tilde{\boldsymbol{V}}_{r,h}(\widehat{\boldsymbol{\theta}})}_{=0} - \tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}^*) = -\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*)\mathbf{D}_h(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*), \quad (A.29)$$

where $\tilde{\boldsymbol{\theta}}^*$ satisfies $\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h \leq \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_h$ with $\|\cdot\|_h$ as defined in Lemma A.2 and

$$\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}) = \sum\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j}\frac{w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)}\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}-\mathbf{v}\|-r)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}) \quad (A.30)$$

By definition, we have that for any $r-h \leq t \leq r+h$,

$$\begin{aligned}|\tilde{g}_{r,h}(t;\boldsymbol{\theta}^*) - \tilde{g}_{r,h}(t;\tilde{\boldsymbol{\theta}}^*)| &= \tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)\left|1 - \exp\left[\theta_0^* - \tilde{\theta}_0^* + h(\theta_1^* - \tilde{\theta}_1^*)\frac{t-r}{h}\cdots + h^p(\theta_p^* - \tilde{\theta}_p^*)\frac{(t-r)^p}{h^p}\right]\right| \\ &\leq \tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)\sqrt{p+1}\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h \exp\left(\sqrt{p+1}\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h\right),\end{aligned}$$

where the last inequality follows form the fact that $|1-e^x| \le |x|e^{|x|}$ and Cauchy-Schwarz inequality. Since $\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h \le \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_h = O_p\left(1/\sqrt{m|D_n|h(r+h)^{d-1}}\right)$ by Lemma A.2, we have that

$$\begin{aligned}
&\eta_{\max}\left[\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] = \sup_{\|\boldsymbol{\delta}\|=1} \boldsymbol{\delta}^T \left[\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] \boldsymbol{\delta} \\
&\quad \le \sup_{\|\boldsymbol{\delta}\|=1} \sum\sum_{i\neq j=1}^{m} \sum_{\mathbf{u}\in X_i} \sum_{\mathbf{v}\in X_j} \frac{w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)} [\boldsymbol{\delta}^T \boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)]^2 \left|\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\tilde{\boldsymbol{\theta}}^*) - \tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\right| \\
&\quad \le \sup_{\|\boldsymbol{\delta}\|=1} \sum\sum_{i\neq j=1}^{m} \sum_{\mathbf{u}\in X_i} \sum_{\mathbf{v}\in X_j} \frac{w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)} [\boldsymbol{\delta}^T \boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)]^2 \tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*) \\
&\qquad\qquad \times \sqrt{p+1}\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h \exp\left(\sqrt{p+1}\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h\right) \\
&\quad = \eta_{\max}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] \times \sqrt{p+1}\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h \exp\left(\sqrt{p+1}\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|_h\right) \\
&\quad = \eta_{\max}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] O_p\left(1/\sqrt{m|D_n|h(r+h)^{d-1}}\right).
\end{aligned}$$

Following exactly the same steps, we can also show that

$$\begin{aligned}
-\eta_{\min}\left[\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] &= \eta_{\max}\left[-\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*) + \tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] \\
&= \eta_{\max}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] O_p\left(1/\sqrt{m|D_n|h(r+h)^{d-1}}\right),
\end{aligned}$$

which implies that

$$\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*) = \eta_{\max}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] O_p\left(1/\sqrt{m|D_n|h(r+h)^{d-1}}\right), \tag{A.31}$$

where the convergence is entry-wise.

The next step is to quantify the variabilities of entries in $\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)$, denoted as H_{ij}'s. Following steps as those in the proof of Lemma A.2 about Var $\left[H''_{m,n}(z_0)\right]$, under conditions

C1, C2(a)-(b), C4 and equation (A.3), some tedious algebra give that

$$
\begin{aligned}
m(m-1)\mathrm{Var}\left(H_{ij}\right) =\ & O(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1| \\
& \qquad \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
& + O(1)\int_{D_n^3} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 + O(1)\int_{D_n^2} w_{r,h}^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
& + mO(1)\int_{D_n^4} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1|d\mathbf{u}_1d\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
& + mO(1)\int_{D_n^3} w_{r,h}(\|\mathbf{u}_1-\mathbf{v}_1\|)w_{r,h}(\|\mathbf{u}_1-\mathbf{u}_2\|)\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
=\ & O(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{w}\|)g(|t-s+w\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w} \\
& + O(1)|D_n|^{-1}\int_{D_n^2} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} + O(1)|D_n|^{-1}\int_{D_n}\left[K_h(\|\mathbf{s}\|-r)\right]^2\mathrm{d}\mathbf{s} \\
& + mO(1)|D_n|^{-1}\int_{D_n^3} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)|g(\|\mathbf{w}\|)-1|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w} \\
& + mO(1)|D_n|^{-1}\int_{D_n^2} K_h(\|\mathbf{s}\|-r)K_h(\|\mathbf{t}\|-r)\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
=\ & O(1)|D_n|^{-1}\int_0^\infty \left[K_h(s-r)\right]^2 s^{d-1}\mathrm{d}s + mO(1)|D_n|^{-1}\left(\int_0^\infty K_h(s-r)s^{d-1}\mathrm{d}s\right)^2 \\
=\ & O(1)|D_n|^{-1}h^{-1}\int_{-r/h}^\infty \left[K(s)\right]^2(r+sh)^{d-1}\mathrm{d}s + mO(1)|D_n|^{-1}\left(\int_{-r/h}^\infty K(s)(r+sh)^{d-1}\mathrm{d}s\right)^2
\end{aligned}
$$

Then, by condition C4, we finally have that

$$
\mathrm{Var}\left(H_{ij}\right) = \frac{(r+h)^{d-1}}{m^2|D_n|h}O(1) + \frac{(r+h)^{2d-2}}{m|D_n|}O(1) \to 0, \text{ as } m|D_n|h\to\infty,
$$

which gives that as $m|D_n|h\to\infty$,

$$
(r+h)^{1-d}\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*) = (r+h)^{1-d}\mathbb{E}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] + o_p(1). \tag{A.32}
$$

Next, we study $\mathbb{E}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right]$. By definition

$$
\mathbb{E}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right] = \int_{D_n^2}\lambda(\mathbf{u})\lambda(\mathbf{v})w_{r,h}(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{A}_h(\|\mathbf{u}-\mathbf{v}\|-r)\boldsymbol{A}_h^T(\|\mathbf{u}-\mathbf{v}\|-r)\tilde{g}_{r,h}(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}.
$$

Recall the definition of $\mathbf{Q}_{n,h}^{(1)}(r)$ in equation (13) and the fact that for any $r-h\le t\le r+h$, $|\tilde{g}_{r,h}(t;\boldsymbol{\theta}^*)-g(r)| = g(r)O(h^{p+1})$ by Lemma A.1, following the similar proof as that of equation (A.31), we have that

$$(r+h)^{1-d}\left\{\mathbb{E}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right]-g(r)\mathbf{Q}^{(1)}_{n,h}(r)\right\}=(r+h)^{1-d}g(r)\eta_{\max}\left[\mathbf{Q}^{(1)}_{n,h}(r)\right]O(h). \quad \text{(A.33)}$$

Using condition C5, equations (A.32)-(A.33) shows that $\eta_{\max}\left[\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)\right]=o_p((r+h)^{d-1})$, which further implies that using (A.31), one has that

$$\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*)=\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)+o_p\left(\frac{\sqrt{(r+h)^{1-d}}}{\sqrt{m|D_n|h}}\right).$$

Furthermore, under condition C5, equations (A.32)-(A.33) also implies that $\tilde{\mathbf{H}}_{h,r}(\boldsymbol{\theta}^*)=g(r)\mathbf{Q}^{(1)}_{n,h}(r)+O_p(h(r+h)^{d-1})+o_p((r+h)^{d-1})$ and hence that

$$\tilde{\mathbf{H}}_{h,r}(\tilde{\boldsymbol{\theta}}^*)=g(r)\mathbf{Q}^{(1)}_{n,h}(r)+O_p(h(r+h)^{d-1})+o_p((r+h)^{d-1})+o_p\left(\frac{\sqrt{(r+h)^{1-d}}}{\sqrt{m|D_n|h}}\right).$$

Plugging the above equality back to equation (A.29), one has that

$$\tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}^*)=\left\{g(r)\mathbf{Q}^{(1)}_{n,h}(r)+o_p((r+h)^{d-1})+o_p\left(\frac{\sqrt{(r+h)^{1-d}}}{\sqrt{m|D_n|h}}\right)\right\}\mathbf{D}_h(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*),$$

which further gives that, under conditions C2(a), C5 and use Lemma A.2, one has

$$\mathbf{D}_h(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)=\left[g(r)\mathbf{Q}^{(1)}_{n,h}(r)\right]^{-1}\left[\tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}^*)+o_p\left(\frac{1}{\sqrt{m|D_n|h(r+h)^{d-1}}}\right)\right]. \quad \text{(A.34)}$$

The proof is completed by observing that the definition of $\mathbf{Z}_1$ and $\mathbf{Z}_2(\boldsymbol{\theta}^*)$ gives that $\tilde{\boldsymbol{V}}_{r,h}(\boldsymbol{\theta}^*)=[\mathbf{Z}_1-\mathbf{Z}_2(\boldsymbol{\theta}^*)]$. □

Proof of Theorem 1. By applying the delta method to $\hat{g}_h(r)=\exp(\hat{\theta}_0)=\exp(\boldsymbol{e}^T\widehat{\boldsymbol{\theta}})$, where $\boldsymbol{e}=(1,0,\ldots,0)^T$, with Lemmas A.4 and A.5, we have that

$$\frac{\sqrt{m|D_n|h}\,[\hat{g}_h(r)-\exp(\theta_0^*)]}{\exp(\theta_0^*)\sqrt{2(m-1+g(r))/(m-1)[g(r)]^{-1}\boldsymbol{e}^T\left[\mathbf{Q}^{(1)}_{n,h}(r)\right]^{-1}\mathbf{Q}^{(2)}_{n,h}(r)\left[\mathbf{Q}^{(1)}_{n,h}(r)\right]^{-1}\boldsymbol{e}}}\xrightarrow{\mathcal{D}}N(0,1).$$

By equation (A.6) in the proof of Lemma A.1, we have that $\exp(\theta_0^*) - g(r) = O(h^{p+1})$. Therefore, it readily follows that

$$\frac{\sqrt{m|D_n|h}\,[\hat{g}_h(r) - g(r)]}{\exp(\theta_0^*)\sqrt{2(m-1+g(r))/(m-1)[g(r)]^{-1}\boldsymbol{e}^T\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\mathbf{Q}_{n,h}^{(2)}(r)\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\boldsymbol{e}}}$$

$$= \frac{\sqrt{m|D_n|h}\,[\hat{g}_h(r) - \exp(\theta_0^*) + \exp(\theta_0^*) - g(r)]}{\sqrt{2(m-1+g(r))/(m-1)g(r)\boldsymbol{e}^T\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\mathbf{Q}_{n,h}^{(2)}(r)\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\boldsymbol{e}}} + o_p(1)$$

$$= \frac{\sqrt{m|D_n|h}\,[\hat{g}_h(r) - \exp(\theta_0^*) + O(h^{p+1})]}{\sqrt{2(m-1+g(r))/(m-1)g(r)\boldsymbol{e}^T\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\mathbf{Q}_{n,h}^{(2)}(r)\left[\mathbf{Q}_{n,h}^{(1)}(r)\right]^{-1}\boldsymbol{e}}} + o_p(1),$$

which completes the proof. □

3 Asymptotic Properties of the Orthogonal Series Estimator

In this Section, we gives detailed proofs of Lemma 3 and Theorem 2.

3.1 Conditions

The following conditions are needed for consistency of the orthogonal series estimator (10).

[C4'] For some $\nu_1 > 0$, the approximation error (14) satisfies (a) $\int_0^R w_o(r)\tilde{\zeta}_L^2(r;\boldsymbol{\theta}_0)\mathrm{d}r = \sum_{l=L+1}^{\infty}\theta_{0,l}^2 = O\left(L^{-2\nu_1}\right)$; (b) $\sup_{0<r\le R}|\tilde{\zeta}_L(r;\boldsymbol{\theta}_0)| = O\left(L^{-\nu_1+\tau_1}\right)$ for some $0 < \tau_1 < \nu_1$; (c) $\sup_{0<r\le R}\|\boldsymbol{\phi}_L(r)\| = O(L^{\nu_2})$ for some $0 \le \nu_2 < \nu_1$; and (d) the weight function is uniformly bounded, i.e., $w_o(r) \le C_w$ for any $0 < r \le R$.

[C5'] As $L \to \infty$, there exist constants c_0, ν_0 where $0 \le 2\nu_0 < \nu_1 - \nu_2$, such that

$$\eta_{\min}\left(\mathbf{Q}_L\right) > c_0 L^{-\nu_0},$$

where $\eta_{\min}(\mathbf{Q})$ denotes the smallest eigenvalue of a matrix $\mathbf{Q}$.

The following additional conditions are needed for asymptotic normality.

[N1'] Either one of the following conditions are true (a) $m \to \infty$; or (b) the mixing coefficient satisfies $\alpha_X(s; 2, \infty) = O(s^{-d-\varepsilon'})$ for some $\varepsilon' > 0$.

[N2'] There exists $\delta' > 2d/\varepsilon'$ such that $\left|g^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k)\right| \leq C_g$ for any $\mathbf{x}_j \in D_n$, $j = 1, \ldots, k$, $k = 2, \ldots, 2(2 + \lceil \delta' \rceil)$.

[N3] For $r \in [0, R]$, define the vector $\boldsymbol{\ell}(r) = (\mathbf{Q}_L)^{-1} \boldsymbol{\phi}_L^T(r)$ and its standardized version $\boldsymbol{\ell}_0(r) = \|\boldsymbol{\ell}(r)\|^{-1}\boldsymbol{\ell}(r)$. Assume that as $m|D_n| \to \infty$ and $L \to \infty$, (a) there exists some constant $c_u > 0$ such that $\boldsymbol{\ell}_0^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}_0(r) \geq c_u$ with $\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*) = \mathrm{Var}\left[\sqrt{m|D_n|}\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*)\right]$; and (b) the basis functions satisfy $\int_0^R \left[w_o(s)|\boldsymbol{\ell}_0^T(r)\boldsymbol{\phi}_L(s)|\right]^{2+\lceil \delta' \rceil} \mathrm{d}s \leq C_\phi$, for some $C_\phi > 0$.

3.2 Sketch of the proof

Step 1 We first derive the asymptotic limit of solutions to $\tilde{\mathbf{U}}_L(\boldsymbol{\theta}) = \mathbf{0}$, namely, $\boldsymbol{\theta}^*$ defined in the (A.35) in the next subsection. As a result, Lemma A.6 gives the asymptotic bias of the orthogonal series estimator of $g(r)$.

Step 2 Lemma A.7 gives the convergence rate of $\widehat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}^*$, which is of the order $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\| = O_p\left(\frac{L^{\nu_2}}{\sqrt{|m|D_n|}}\right)$;

Step 3 Find the asymptotic normality of $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*$ through Lemmas A.8 to A.9, following the approach proposed in Biscio and Waagepetersen (2019) .

3.3 The asymptotic bias

Let $\boldsymbol{\theta}_0 = (\theta_{0,1}, \ldots, \theta_{0,L})$, where $\theta_{0,l}$'s are the first L coefficients of the orthogonal series expansion of $g(r)$ with respect to the basis functions $\phi_l(r)$'s and suppose there exists a vector $\boldsymbol{\theta}^* = (\theta_1^*, \ldots, \theta_L^*)^T$ such that

$$\int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)\left[g(\|\mathbf{u}-\mathbf{v}\|) - \tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\right]\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}|)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} = \mathbf{0}. \quad \text{(A.35)}$$

The following Lemma quantifies the distance between $g(r)$ and $\tilde{g}_L(r;\boldsymbol{\theta}^*)$.

Lemma A.6. *Under conditions C1-C3 and C4'-C5', we have that as* $L \to \infty$,

$$\begin{aligned} \|\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*\| &= O(L^{\nu_0-\nu_1}), && \text{(A.36)}\\ \sup_{0<r<R} |g(r) - \tilde{g}_L(r;\boldsymbol{\theta}^*)| &= O\left(L^{-\nu_1+\max\{\tau_1,\nu_0+\nu_2\}}\right) = o(1), && \text{(A.37)}\\ \sup_{0<r<R} |\tilde{g}_L(r;\boldsymbol{\theta}^*)| &= O(1), && \text{(A.38)} \end{aligned}$$

where ν_0, ν_1, τ_1 *and* ν_2 *are defined in conditions C4' and C5'.*

Proof. It is straightforward to see that by definition, $\boldsymbol{\theta}^*$ is the solution to (A.35), which also maximizes the following target function

$$\ell(\boldsymbol{\theta}) = \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)\left\{g(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|) - \exp\left[\boldsymbol{\theta}^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\right]\right\}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}.$$

Let $\boldsymbol{\nu}_n$ be an arbitrary sequence on the sphere $\{\boldsymbol{\nu} \in \mathbb{R}^L : \|\boldsymbol{\nu}\| = L^{-\nu_1+\nu_0}\}$ and define functions $\Delta_n(r) = \boldsymbol{\nu}_n^T\boldsymbol{\phi}_L(r)$. Let $f_0(r) = \boldsymbol{\theta}_0^T\boldsymbol{\phi}_L(r)$, $0 < r < R$, and define the function of a scalar z as follows

$$\begin{aligned} h(z) = &\int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)\\ &\times\left\{g(\|\mathbf{u}-\mathbf{v}\|)\left[f_0(\|\mathbf{u}-\mathbf{v}\|) + z\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\right] - \exp\left[f_0(\|\mathbf{u}-\mathbf{v}\|) + z\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\right]\right\}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}. \end{aligned}$$

We shall show that for any $z_0 > 0$, $h'(z_0) < 0$ and $h'(-z_0) > 0$. This implies that the maximum of $\ell(\boldsymbol{\theta})$, namely $\boldsymbol{\theta}^*$, satisfies $f_0(r) - z_0\Delta_n(r) \le \boldsymbol{\theta}^{*T}\boldsymbol{\phi}_L(r) \le f_0(r) + z_0\Delta_n(r)$, using the fact that $\ell(\boldsymbol{\theta})$ is a concave function of $\boldsymbol{\theta}$. Some straightforward calculus gives that

$$\begin{aligned}
h'(z) &= \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\Delta_n(\|\mathbf{u}-\mathbf{v}\|) \\
&\qquad \times \left\{1-\exp\left[-\tilde{\zeta}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}_0)+z\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\right]\right\}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&= \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\left\{1-\exp\left[-\tilde{\zeta}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}_0)\right]\right\}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&+ \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\exp\left[-\tilde{\zeta}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}_0)\right] \\
&\qquad \times \left\{1-\exp\left[z\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\right]\right\}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&= \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\tilde{\zeta}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}_0)\left[1+o(1)\right]\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&+ \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\left[-z\Delta_n(\|\mathbf{u}-\mathbf{v}\|)\right]\left[1+o(1)\right]\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v},
\end{aligned}$$

where $\tilde{\zeta}_L(r;\boldsymbol{\theta}_0)$ is the approximation error defined in equation (13). The last equation follows from the Taylor expansion $1-e^x=-x\left[1+e^{x^*}x/2\right]$, for some $|x^*|<|x|$, and the condition C4', which ensures that as $L\to\infty$,

$$\begin{aligned}
&\sup_{0<r<R}|\zeta_L(r;\boldsymbol{\theta}_0)| = O(L^{\tau_1-\nu_1}) = o(1), \\
&\sup_{0<r<R}|\Delta_n(r)| \le \|\boldsymbol{\nu}_n\|\sup_{0<r<R}\|\boldsymbol{\phi}_L(r)\| = O(L^{\nu_0+\nu_2-\nu_1}) = o(1).
\end{aligned}$$

When $z>0$, using conditions C1-C3, C4' and the Hölder's inequality, we can derive that

$$\begin{aligned}
h'(z) &\le O(1)\int_0^R w_o(s)g(s)\Delta_n(s)\tilde{\zeta}_L(s;\boldsymbol{\theta}_0)s^{d-1}\mathrm{d}s \\
&\quad - z\left[1+o(1)\right]\boldsymbol{\nu}_n^T\underbrace{\left\{\int_{D_n^2}\lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L^T(\|\mathbf{u}-\mathbf{v}\|)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}\right\}}_{Q_L \text{ defined in (13)}}\boldsymbol{\nu}_n \\
&\le O(1)\sqrt{\int_0^R w_o(s)\Delta_n^2(s)s^{d-1}\mathrm{d}s}\sqrt{\int_0^R w_o(s)\tilde{\zeta}_L^2(s;\boldsymbol{\theta}_0)\mathrm{d}s} - z\left[1+o(1)\right]\|\boldsymbol{\nu}_n\|^2\times\eta_{\min}\left[Q_L\right] \\
&\le O(1)\sqrt{\|\boldsymbol{\nu}_n\|^2\int_0^R w_o(s)\|\boldsymbol{\phi}_L(s)\|^2 s^{d-1}\mathrm{d}s}\sqrt{\int_0^R w_o(s)\tilde{\zeta}_L^2(s;\boldsymbol{\theta}_0)\mathrm{d}s} - z\left[1+o(1)\right]\|\boldsymbol{\nu}_n\|^2\times\eta_{\min}\left[Q_L\right] \\
&= O\left(L^{-\nu_1}\right)\|\boldsymbol{\nu}_n\| - z\left[1+o(1)\right]\|\boldsymbol{\nu}_n\|^2\times\eta_{\min}\left[Q_L\right].
\end{aligned}$$

Finally, under condition C5, $\|\boldsymbol{\nu}_n\| = L^{-\nu_1+\nu_0}$ is sufficient to ensure that there exists a $z_0 > 0$ such that $h'(z_0) < 0$.

Similarly, $\|\boldsymbol{\nu}_n\| = L^{-\nu_1+\nu_0}$ is sufficient to ensure that

$$\begin{aligned}
h'(-z_0) &\geq -O(1) \int_0^R w_o(s) g(s) |\Delta_n(s)| |\tilde{\zeta}_L(s; \boldsymbol{\theta}_0)| s^{d-1} \mathrm{d}s \\
&\quad + z_0 \left[1 + o(1)\right] \boldsymbol{\nu}_n^T \underbrace{\left\{ \int_{D_n^2} \lambda(\mathbf{u}) \lambda(\mathbf{v}) w_R(\|\mathbf{u}-\mathbf{v}\|) g(\|\mathbf{u}-\mathbf{v}\|) \boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|) \boldsymbol{\phi}_L^T(\|\mathbf{u}-\mathbf{v}\|) \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \right\}}_{Q_L} \boldsymbol{\nu}_n \\
&\geq -O(1) \sqrt{\int_0^R w_o(s) \Delta_n^2(s) s^{d-1} \mathrm{d}s} \sqrt{\int_0^R w_o(s) \tilde{\zeta}_L^2(s; \boldsymbol{\theta}_0) \mathrm{d}s} + z_0 \left[1 + o(1)\right] \|\boldsymbol{\nu}_n\|^2 \times \eta_{\min}\left[Q_L\right] \\
&= -O\left(L^{-\nu_1}\right) \|\boldsymbol{\nu}_n\| + z_0 \left[1 + o(1)\right] \|\boldsymbol{\nu}_n\|^2 \times \eta_{\min}\left[Q_L\right] > 0.
\end{aligned}$$

Therefore, we have shown that $\boldsymbol{\theta}^{*T} \boldsymbol{\phi}_L(r)$ is between $\boldsymbol{\theta}_0^T \boldsymbol{\phi}_L(s) \pm z_0 \boldsymbol{\nu}_n^T \boldsymbol{\phi}_L(s)$, and hence

$$\begin{aligned}
\|\boldsymbol{\theta}^* - \boldsymbol{\theta}_0\|^2 &= (\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)^T \left[\int_0^R w_o(s) \boldsymbol{\phi}_L(s) \boldsymbol{\phi}_L^T(s) \mathrm{d}s\right] (\boldsymbol{\theta}^* - \boldsymbol{\theta}_0) \\
&= \int_0^R w_o(s) \left[\boldsymbol{\theta}^{*T} \boldsymbol{\phi}_L(s) - \boldsymbol{\theta}_0^T \boldsymbol{\phi}_L(s)\right]^2 \mathrm{d}s \leq z_0^2 \int_0^R w_o(s) \left[\boldsymbol{\nu}_n^T \boldsymbol{\phi}_L(s)\right]^2 \mathrm{d}s = z_0^2 \|\boldsymbol{\nu}_n\|^2,
\end{aligned}$$

which completes the proof of equation (A.36).

Furthermore, to show (A.37), note that, under condition C4'(b),

$$|g(r) - \tilde{g}_L(r; \boldsymbol{\theta}_0)| = g(r) \left| 1 - \exp\left[-\sum_{l=L+1}^{\infty} \theta_{0,l} \phi_l(r) \right] \right| = g(r) O(L^{-\nu_1+\tau_1}) = O(L^{-\nu_1+\tau_1}).$$

Under condition C2(a), the above result also implies that $\sup_{0<r<R} \tilde{g}_L(r; \boldsymbol{\theta}_0) = O(1)$. Then, we have that

$$\begin{aligned}
|g(r) - \tilde{g}_L(r; \boldsymbol{\theta}^*)| &\leq |g(r) - \tilde{g}_L(r; \boldsymbol{\theta}_0)| + |\tilde{g}_L(r; \boldsymbol{\theta}_0) - \tilde{g}_L(r; \boldsymbol{\theta}^*)| \\
&= O(L^{-\nu_1+\tau_1}) + \tilde{g}_L(r; \boldsymbol{\theta}_0) \left| 1 - \exp\left[\sum_{l=1}^{L} (\theta_l^* - \theta_{0,l}) \phi_l(r) \right] \right| \\
&= O(L^{-\nu_1+\tau_1}) + \tilde{g}_L(r; \boldsymbol{\theta}_0) O\left(\sup_{0<r\leq R} \|\boldsymbol{\phi}_L(r)\| \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\| \right) \\
&= O(L^{-\nu_1+\tau_1}) + O(L^{\nu_0-\nu_1+\nu_2}) \\
&= o(1),
\end{aligned}$$

where the last equality follows from condition C5', where we have assumed that $0 \leq 2\nu_0 < \nu_1 - \nu_2$. Equation (A.37) immediately follows by noting that all the upper bounds do not depend on r. Equation (A.38) is trivial by combining equation (A.37) and condition C2(a).

□

3.4 Proof of Lemma 3

Lemma A.7. *Under conditions C1-C3, and C4'-C5', we have that as* $L \to \infty$ *and* $L^{4\nu_0+2\nu_2}/m|D_n| \to 0$,

$$\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\| \;=\; O_p\left(\frac{L^{\nu_0}}{\sqrt{m|D_n|}}\right) \tag{A.39}$$

where $\boldsymbol{\theta}^*$ *is defined in equation* (A.35).

Proof. It is straightforward to see that solving estimating equation (9), i.e., $\tilde{\mathbf{U}}_L(\boldsymbol{\theta}) = \mathbf{0}$, is equivalent to maximizing the following composite log likelihood function

$$\begin{aligned}\tilde{L}(\boldsymbol{\theta}) = \frac{1}{m}\sum_{i=1}^{m}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_R(\|\mathbf{u}-\mathbf{v}\|)\log\left[\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta})\right] \\ - \frac{1}{m(m-1)}\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} w_R(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}),\end{aligned} \tag{A.40}$$

with respect to $\boldsymbol{\theta}$, because $\partial\tilde{L}(\boldsymbol{\theta})/\partial\boldsymbol{\theta} = \tilde{\mathbf{U}}_L(\boldsymbol{\theta})/m$. Note that the Hessian matrix

$$\tilde{\mathbf{H}}_L(\boldsymbol{\theta}) = \frac{\partial^2\tilde{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T} = -\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j}\frac{w_R(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)}\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta})\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L^T(\|\mathbf{u}-\mathbf{v}\|)$$

is negative definitive, which implies that $\tilde{L}(\boldsymbol{\theta})$ is a concave function of $\boldsymbol{\theta}$.

We use the same steps as in the proof of Lemma A.2. Let $J_{m,n}$ be a sequence of positive real numbers such that $J_{m,n} \to \infty$ as $m \to \infty$ and/or $n \to \infty$. We shall first show that for any given $\varepsilon > 0$ there exists a large constant C_ϵ such that, for large m or/and n,

$$\mathbb{P}\left\{\sup_{\|\boldsymbol{\delta}_L\|=1}\tilde{L}(\boldsymbol{\theta}^* + C_\epsilon J_{m,n}^{-1/2}\boldsymbol{\delta}_L) < \tilde{L}(\boldsymbol{\theta}^*)\right\} \geq 1 - \varepsilon. \tag{A.41}$$

Inequality (A.41) implies that with probability tending to 1 the function $\tilde{L}(\boldsymbol{\theta})$ has a local maximum, denoted as $\widehat{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}}_{m,n}$, in the ball $\boldsymbol{\Theta}_{m,n} = \{\boldsymbol{\theta}^* + J_{m,n}^{-1/2} C \boldsymbol{\delta}_L : \|\boldsymbol{\delta}_L\| = 1\}$. It then follows that $J_{m,n}\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2$ is bounded in probability; i.e., $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2 = O_p(J_{m,n}^{-1})$.

To show (A.41), for any fixed $\boldsymbol{\delta}_L \in \mathbb{R}^L$ that $\|\boldsymbol{\delta}_L\| = 1$ define a function of $z > 0$ as

$$H_{m,n}(z) = -\tilde{L}(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_L). \tag{A.42}$$

Then

$$H'_{m,n}(z) = -\frac{J_{m,n}^{-1/2}}{m} \boldsymbol{\delta}_L^T \tilde{\mathbf{U}}_L(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_L), \tag{A.43}$$

$$H''_{m,n}(z) = -J_{m,n}^{-1} \boldsymbol{\delta}_L^T \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_L) \boldsymbol{\delta}_L \tag{A.44}$$

and by the Taylor's theorem

$$H_{m,n}(z) = H_{m,n}(0) + H'_{m,n}(0) z + H''_{m,n}(t_z) \frac{z^2}{2}$$

for any $z > 0$ and some $0 < t_z < z$, which implies that

$$\tilde{L}(\boldsymbol{\theta}^* + z J_{m,n}^{-1/2} \boldsymbol{\delta}_L) - \tilde{L}(\boldsymbol{\theta}^*) = H_{m,n}(0) - H_{m,n}(z) = -z \left[H'_{m,n}(0) + \frac{z}{2} H''_{m,n}(t_z) \right].$$

By definition, $H_{m,n}(z)$ is a convex function of z since $H''_{m,n}(z) \geq 0$ for any constant z. Therefore, to find a large enough C_ϵ so that (A.41) holds, it suffices to show that $H'_{m,n}(0) = O_p\left[H''_{m,n}(t_z)\right]$ for any $z > 0$. We first investigate $H'_{m,n}(0)$. By the definition of $\boldsymbol{\theta}^*$ in (A.35), we have that $\mathbb{E}\left[H'_{m,n}(0)\right] = 0$. Furthermore, similarly as in the proof of Lemma A.2 the variance can be shown as

$$
\begin{aligned}
mJ_{m,n}\mathrm{Var}\left[H'_{m,n}(0)\right] = &\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)\big[g^{(4)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2,\mathbf{v}_2) \\
&- g(\|\mathbf{u}_1-\mathbf{v}_1\|)g(\|\mathbf{u}_2-\mathbf{v}_2\|)\big] \times \left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&- 4\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)[g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2) - g(\|\mathbf{u}_1-\mathbf{v}_1\|)] \\
&\times \tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&+ \frac{2}{m-1}\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*) \\
&\times [g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&+ \frac{4(m-2)}{m-1}\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1] \\
&\times \tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
&+ 4\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2) \\
&\times \left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&- 8\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)[g(\|\mathbf{u}_1-\mathbf{v}_1\|) - \tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)] \\
&\times \tilde{g}_L(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&+ \frac{4}{m-1}\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*) \\
&\times \tilde{g}_L(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)[g(\|\mathbf{v}_1-\mathbf{u}_2\|)-1]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&+ 2\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\left[w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]^2 g(\|\mathbf{u}_1-\mathbf{v}_1\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
&+ \frac{2}{m-1}\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_R^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_L^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1.
\end{aligned}
$$

Therefore, under conditions C1-C3 and equations (A.37)-(A.38), we can further simplify $mJ_{m,n}\mathrm{Var}\left[H'_{m,n}(0)\right]$ as follows

$$
\begin{aligned}
mJ_{m,n}\mathrm{Var}\left[H'_{m,n}(0)\right] &= O(1)\int_{D_n^4} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)\big[g^{(4)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2,\mathbf{v}_2)\\
&\quad - g(\|\mathbf{u}_1-\mathbf{v}_1\|)g(\|\mathbf{u}_2-\mathbf{v}_2\|)\big]\times\left|\boldsymbol{\phi}_L^T(\|\mathbf{u}_1-\mathbf{v}_1\|)\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{u}_2-\mathbf{v}_2\|)\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&\quad + O(1)\int_{D_n^4} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)[g^{(3)}(\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2)-g(\|\mathbf{u}_1-\mathbf{v}_1\|)]\\
&\qquad \times\left|\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&\quad + O(m^{-1})\int_{D_n^4} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1]\\
&\qquad \times\left|\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2\\
&\quad + O(1)\int_{D_n^4} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1]\\
&\quad \times\left|\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2\\
&\quad + O(1)\int_{D_n^3} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\left|\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\\
&\quad + O(1)\int_{D_n^2} [w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)]^2\left|\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right|^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\\
&= O(1)|D_n|\int_{D_n^3} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)\left|g^{(4)}(\mathbf{w}\mathbf{t}+\mathbf{w},\mathbf{w})-g(\|\mathbf{s}\|)g(\|\mathbf{t}\|)\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&\quad + O(1)|D_n|\int_{D_n^3} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)|g^{(3)}(\mathbf{s},\mathbf{w})-g(\|\mathbf{s}\|)|\left|\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&\quad + \frac{1}{m}O(1)|D_n|\int_{D_n^3} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)|g(\|\mathbf{w}\|)g(\|\mathbf{t}-\mathbf{s}+\mathbf{w}\|)-1|\left|\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&\quad + O(1)|D_n|\int_{D_n^3} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)|g(\|\mathbf{w}\|)-1|\left|\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w}\\
&\quad + O(1)|D_n|\int_{D_n^2} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)\left|\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right|\left|\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right|\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\\
&\quad + O(1)|D_n|\int_{D_n} [w_R(\|\mathbf{s}\|)]^2\left|\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right|^2\mathrm{d}s\\
&= O(1)|D_n|\left(\int_0^\infty w_R(s)\left|\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right|s^{d-1}\mathrm{d}s\right)^2 + O(1)|D_n|\int_0^\infty\left[w_R(s)\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s,
\end{aligned}
$$

where the last equality follows from conditions C2-C3 and equations (A.37)-(A.38). Recall that by definition of orthogonal basis functions, we have that $\int_0^R w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2\mathrm{d}s =$

$\boldsymbol{\delta}_L^T\boldsymbol{\delta}_L = 1$. Finally, by the condition C4, we have that

$$\begin{aligned} mJ_{m,n}\mathrm{Var}\left[H'_{m,n}(0)\right] &= O(1)|D_n|^{-1}\left\{\int_0^R w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]s^{d-1}\mathrm{d}s\right\}^2 \\ &\quad + O(1)|D_n|^{-1}\int_0^R w_o^2(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s \\ &\le O(1)|D_n|^{-1}\left\{\int_0^R w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2\mathrm{d}s\right\}\int_0^R w_o(s)s^{2(d-1)}\mathrm{d}s \\ &\quad + O(1)|D_n|^{-1}\int_0^R w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2\mathrm{d}s = O\left(\frac{1}{|D_n|}\right), \end{aligned}$$

where the second last inequality follows from the Hölder's inequality.

Combing with the fact that $\mathbb{E}\left[H'_{m,n}(0)\right] = 0$, we have that

$$H'_{m,n}(0) = O_P\left(\frac{1}{\sqrt{m|D_n|J_{m,n}}}\right). \tag{A.45}$$

Now we proceed to study $H''_{m,n}(t_z)$. Some tedious algebra gives that

$$\begin{aligned} &J_{m,n}^2 m(m-1)\mathrm{Var}\left[H''_{m,n}(t_z)\right] \\ &= 2\int_{D_n^4}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)\tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\tilde{\boldsymbol{\theta}}^*) \\ &\qquad \times\left[g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\ &\quad + 4\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) \\ &\qquad \times\tilde{g}_L(\|\mathbf{u}_1-\mathbf{u}_2\|;\tilde{\boldsymbol{\theta}}^*)g(\|\mathbf{v}_1-\mathbf{u}_2\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\ &\quad + 2\int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_R^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_L^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)g(\|\mathbf{u}_1-\mathbf{v}_1\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^4\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\ &\quad + 4(m-2)\int_{D_n^4}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1] \\ &\qquad \times\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)\tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\tilde{\boldsymbol{\theta}}^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\ &\quad + 4(m-2)\int_{D_n^3}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{u}_2\|;\tilde{\boldsymbol{\theta}}^*) \\ &\qquad \times\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2, \end{aligned}$$

where $\tilde{\boldsymbol{\theta}}^* = \boldsymbol{\theta}^* + t_z J_{m,n}^{-1/2}\boldsymbol{\delta}_L$.

Using equations (A.37)-(A.38), under conditions C1-C3, we can further simplify $J_{m,n}^2 m(m-1)\mathrm{Var}\left[H''_{m,n}(t_z)\right]$ as follows

$$
\begin{aligned}
J_{m,n}^2 m(m-1)\mathrm{Var}\left[H''_{m,n}(t_z)\right] &= O(1)\int_{D_n^4} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1| \\
&\qquad \times \left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&+ O(1)\int_{D_n^3} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&+ O(1)\int_{D_n^2} w_R^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^4 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
&+ mO(1)\int_{D_n^4} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)|g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1| \\
&\qquad \times \left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
&+ mO(1)\int_{D_n^3} w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&= O(1)|D_n|\int_{D_n^3} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)|g(\|\mathbf{w}\|)g(\|\mathbf{t}-\mathbf{s}+\mathbf{w}\|)-1|\left[\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w} \\
&+ O(1)|D_n|\int_{D_n^2} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)\left[\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
&+ O(1)|D_n|\int_{D_n} [w_R(\|\mathbf{s}\|)]^2\left[\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right]^4 \mathrm{d}s \\
&+ mO(1)|D_n|\int_{D_n^3} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)|g(\|\mathbf{w}\|)-1|\left[\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{w} \\
&+ mO(1)|D_n|\int_{D_n^2} w_R(\|\mathbf{s}\|)w_R(\|\mathbf{t}\|)\left[\boldsymbol{\phi}_L^T(\|\mathbf{s}\|)\boldsymbol{\delta}_L\right]^2\left[\boldsymbol{\phi}_L^T(\|\mathbf{t}\|)\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
&= O(1)|D_n|\int_0^\infty [w_R(s)]^2\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^4 s^{d-1}\mathrm{d}s + mO(1)|D_n|\left(\int_0^\infty w_R(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s\right)^2.
\end{aligned}
$$

Recall that by definition of orthogonal basis functions, we have that $\int_0^R w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2 \mathrm{d}s = \boldsymbol{\delta}_L^T\boldsymbol{\delta}_L = 1$. Then, by the condition C4', we have that

$$
\begin{aligned}
J_{m,n}^2 m(m-1)\mathrm{Var}\left[H''_{m,n}(t_z)\right] &= O(1)|D_n|^{-1}\int_0^R w_o^2(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^4 s^{d-1}\mathrm{d}s \\
&\quad + O(1)m|D_n|^{-1}\left\{\int_0^\infty w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s\right\}^2 \\
&\le O(1)|D_n|^{-1}\sup_{0<r\le R}\|\boldsymbol{\phi}_L(r)\|^2 \times \int_0^R w_o(s)\left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}_L\right]^2 \mathrm{d}s + O(1)m|D_n|^{-1} \\
&= O(L^{2\nu_2}|D_n|^{-1}) + O(m|D_n|^{-1}),
\end{aligned}
$$

which immediately implies that

$$\text{Var}\left[H''_{m,n}(t_z)\right] = O\left(\frac{L^{2\nu_2}}{J_{m,n}^2 m^2 |D_n|}\right) + O\left(\frac{1}{J_{m,n}^2 m|D_n|}\right). \tag{A.46}$$

On the other hand, we have that

$$\mathbb{E}\left[H''_m(t_z)\right] = J_{m,n}^{-1}\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}_m^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2 \mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1.$$

Using definition of $\mathbf{Q}_L$ in condition C6 and the fact that $\eta_{\min}\left[\mathbf{Q}_L\right] = \inf_{\|\boldsymbol{\eta}\|^2=1}\boldsymbol{\eta}^T\mathbf{Q}_L\boldsymbol{\eta}$, we have that

$$\begin{aligned}
\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] - \eta_{\min}\left[\mathbf{Q}_{n,h}\right] &\geq \int_{D_n^2}\lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|) \\
&\quad\times\left[\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\tilde{\boldsymbol{\theta}}^*) - g(\|\mathbf{u}_1-\mathbf{v}_1\|)\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
&= O(1)|D_n|\int_0^\infty w_R(s)\left|\tilde{g}_L(s;\tilde{\boldsymbol{\theta}}^*) - g(s)\right|\left[\boldsymbol{\phi}_L(s)^T\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s \\
&= O(1)\int_0^R w_o(s)\left|\tilde{g}_L(s;\tilde{\boldsymbol{\theta}}^*) - g(s)\right|\left[\boldsymbol{\phi}_L(s)^T\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s \\
&= O(1)\sup_{0<r\leq R}\|\boldsymbol{\phi}_L(r)\|\int_0^R w_o(s)\left|\tilde{g}_L(s;\tilde{\boldsymbol{\theta}}^*) - g(s)\right|\left|\boldsymbol{\phi}_L(s)^T\boldsymbol{\delta}_L\right|\mathrm{d}s.
\end{aligned} \tag{A.47}$$

since $\tilde{\boldsymbol{\theta}}^* = \boldsymbol{\theta}^* + t_z J_{m,n}^{-1/2}\boldsymbol{\delta}_L$, it is straightforward to show that, under conditions C4' and provided the $\sup_{0<r\leq R}\left|J_{m,n}^{-1/2}\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(r)\right| = O(1)$,

$$\begin{aligned}
\left|\tilde{g}_L(r;\tilde{\boldsymbol{\theta}}^*) - g(r)\right| &= g(r)\left|1 - \exp\left[(\boldsymbol{\theta}^*-\boldsymbol{\theta}_0)^T\boldsymbol{\phi}_L(r) + t_z J_{m,n}^{-1/2}\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(r) - \tilde{\zeta}_L(r;\boldsymbol{\theta}_0)\right]\right| \\
&= O(1)\left\{\left|(\boldsymbol{\theta}^*-\boldsymbol{\theta}_0)^T\boldsymbol{\phi}_L(r)\right| + \left|t_z J_{m,n}^{-1/2}\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(r)\right| + \left|\tilde{\zeta}_L(r;\boldsymbol{\theta}_0)\right|\right\},
\end{aligned}$$

which further gives that, under condition C4' and using (A.36) in Lemma A.6,

$$\begin{aligned}
\int_0^R w_o(s)\left|\tilde{g}_L(s;\tilde{\boldsymbol{\theta}}_m^*) - g(s)\right|\left|\boldsymbol{\phi}_L(|s|)^T\boldsymbol{\delta}_L\right|\mathrm{d}s &= O(1)\int_0^R w_o(s)\left|(\boldsymbol{\theta}^*-\boldsymbol{\theta}_0)^T\boldsymbol{\phi}_L(r)\right|\left|\boldsymbol{\phi}_L(|s|)^T\boldsymbol{\delta}_L\right|\mathrm{d}s \\
&\quad + O(1)\int_0^R w_o(s)\left|t_z J_{m,n}^{-1/2}\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(r)\right|\left|\boldsymbol{\phi}_L(|s|)^T\boldsymbol{\delta}_L\right|\mathrm{d}s \\
&\quad + O(1)\int_0^R w_o(s)\left|\tilde{\zeta}_L(r;\boldsymbol{\theta}_0)\right|\left|\boldsymbol{\phi}_L(|s|)^T\boldsymbol{\delta}_L\right|\mathrm{d}s \\
&= O(1)\left[\int_0^R w_o(s)\left|(\boldsymbol{\theta}^*-\boldsymbol{\theta}_0)^T\boldsymbol{\phi}_L(r)\right|^2\mathrm{d}s\right]^{1/2} + O(1)z_0 J_{m,n}^{-1/2} + O(1)\left[\int_0^R w_o(s)\tilde{\zeta}_L^2(r;\boldsymbol{\theta}_0)\mathrm{d}s\right]^{1/2} \\
&= O(\|\boldsymbol{\theta}^*-\boldsymbol{\theta}_0\| + J_{m,n}^{-1/2} + L^{-\nu_1}) \\
&= O(L^{\nu_0-\nu_1} + J_{m,n}^{-1/2} + L^{-\nu_1}) = O(L^{\nu_0-\nu_1} + J_{m,n}^{-1/2}).
\end{aligned}$$

Combining the above result, equation (A.47) and condition C2, we have that if $J_{m,n}^{-1/2}L^{\nu_2} = O(1)$,

$$\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] - \eta_{\min}\left[\mathbf{Q}_{n,h}\right] = O(L^{\nu_2+\nu_0-\nu_1} + J_{m,n}^{-1/2}L^{\nu_2})$$

By condition C2(a) and C5', the above equation gives that

$$L^{\nu_0}\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] \geq c_0 + O(L^{\nu_2+2\nu_0-\nu_1} + J_{m,n}^{-1/2}L^{\nu_0+\nu_2}). \tag{A.48}$$

Hence for the constant $c = c_0$, we have that

$$\begin{aligned}
&\mathbb{P}\left(L^{\nu_0}J_{m,n}H''_m(t_z) < \frac{c}{2}\right) = \mathbb{P}\left\{J_{m,n}H''_m(t_z) - \mathbb{E}\left[J_{m,n}H''_m(t_z)\right] < c/2L^{-\nu_0} - \mathbb{E}\left[J_{m,n}H''_m(t_z)\right]\right\} \\
&\quad \leq \mathbb{P}\left\{\left|J_{m,n}H''_m(t_z) - \mathbb{E}\left[J_{m,n}H''_m(t_z)\right]\right| > \left|c/2L^{-\nu_0} - \mathbb{E}\left[J_{m,n}H''_m(t_z)\right]\right|\right\} \\
&\qquad \times I\left\{\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] > c/2L^{-\nu_0}\right\} + I\left\{\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] \leq c/2L^{-\nu_0}\right\} \\
&\quad \leq \frac{\mathrm{Var}\left[J_{m,n}H''_{m,n}(t_z)\right]}{\left|c/2L^{-\nu_0} - \mathbb{E}\left[J_{m,n}H''_m(t_z)\right]\right|^2} I\left\{\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] > c/2L^{-\nu_0}\right\} + I\left\{\mathbb{E}\left[J_{m,n}H''_m(t_z)\right] \leq c/2L^{-\nu_0}\right\} \\
&\quad = O\left(\frac{L^{2\nu_2+2\nu_0}}{m^2|D_n|}\right) + O\left(\frac{L^{2\nu_0}}{m|D_n|}\right) + o(1),
\end{aligned}$$

where the last equality follows from equations (A.46) and (A.48) when $J_{m,n} \to \infty$ and $L^{2\nu_0+2\nu_2}/J_{m,n} \to 0$. Therefore, as long as $\frac{L^{2\nu_0+2\nu_2}}{m^2|D_n|} + \frac{L^{2\nu_0}}{m|D_n|} \to 0$, $J_{m,n} \to \infty$ and $L^{2\nu_0+2\nu_2}/J_{m,n} \to 0$, we have that

$$\mathbb{P}\left(J_{m,n}H''_m(t_z) \geq c_0L^{-\nu_0}/2\right) \to 1, \tag{A.49}$$

where c_0 is the constant defined in condition C5'.

We have already shown in equation (A.45) that

$$H'_{m,n}(0) = O_P\left(\frac{1}{\sqrt{m|D_n|J_{m,n}}}\right).$$

hence as long as $\frac{J_{m,n}L^{2\nu_0}}{m|D_n|} = O(1)$, we have that $H'_{m,n}(0) = O_P(H''_{m,n}(t_z))$. In other words, for any sequence $J_{m,n}$ satisfying $\frac{J_{m,n}L^{2\nu_0}}{m|D_n|} \to 0$ and $L^{2\nu_0+2\nu_2}/J_{m,n} \to 0$, the right hand side of the inequality

$$\mathbb{P}\left\{|H'_{m,n}(0)| \geq \frac{z}{2}H''_{m,n}(t_z)\right\} \leq \mathbb{P}\left\{J_{m,n}H''_{m,n}(t_z) \leq \frac{c_0}{2}\right\} + \mathbb{P}\left\{J_{m,n}|H'_{m,n}(0)| \geq \frac{zc_0}{4}\right\}$$

can be arbitrary small by choosing z and m and/or n large enough. Therefore, for any given $\epsilon > 0$, there exists $z_\epsilon > 0$ such that for large m and/or n,

$$\mathbb{P}\left\{\tilde{L}(\boldsymbol{\theta}^* + z_\epsilon J_{m,n}^{-1/2}\boldsymbol{\delta}_L) < \tilde{L}(\boldsymbol{\theta}^*)\right\} = \mathbb{P}\left\{z_\epsilon H'_{m,n}(0) + \frac{z_\epsilon^2}{2} H''_{m,n}(t_{z_\epsilon}) > 0\right\} \geq 1 - \epsilon.$$

Thus, (A.41) holds, which completes the proof of equation (A.39). □

Proof of Lemma 3. Our goal is to show

$$\sup_{0<r<R}\left|g(r) - \tilde{g}_L(r;\widehat{\boldsymbol{\theta}})\right| = O\left(L^{-\nu_1+\max\{\tau_1,\nu_0+\nu_2\}}\right) + O_p\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right). \quad \text{(A.50)}$$

To show (A.50), using equations (A.37)-(A.38), we have that

$$\begin{aligned}
\left|g(r) - \tilde{g}_L(r;\widehat{\boldsymbol{\theta}})\right| &\leq |g(r) - \tilde{g}_L(r;\boldsymbol{\theta}^*)| + \left|\tilde{g}_L(r;\boldsymbol{\theta}^*) - \tilde{g}_L(r;\widehat{\boldsymbol{\theta}})\right| \\
&= O\left(L^{-\nu_1+\max\{\tau_1,\nu_0+\nu_2\}}\right) + \tilde{g}_L(r;\boldsymbol{\theta}^*)\left|1 - \exp\left[\sum_{l=1}^{L}(\hat{\theta}_l - \theta_l^*)\phi_l(r)\right]\right| \\
&= O\left(L^{-\nu_1+\max\{\tau_1,\nu_0+\nu_2\}}\right) + \tilde{g}_L(r;\boldsymbol{\theta}^*)O\left(\sup_{0<r\leq R}\|\boldsymbol{\phi}_L(r)\|\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|\right) \\
&= O\left(L^{-\nu_1+\max\{\tau_1,\nu_0+\nu_2\}}\right) + O_p\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right) \\
&= o(1),
\end{aligned}$$

where the upper bounds does not depend on r, which completes the proof. □

3.5 Proof of Theorem 2

Lemma A.8. *Let $\tilde{\sigma}^2_{\boldsymbol{\delta}}(\boldsymbol{\theta}^*) = \boldsymbol{\delta}_L^T\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\delta}_L^T$ with $\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*) = \mathrm{Var}\left[\sqrt{m|D_n|}\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)\right]$. If the vector $\boldsymbol{\delta}_L$ satisfies (a) $\|\boldsymbol{\delta}_L\| = 1$; (b) $\int_0^R\left[w_o(s)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(s)|\right]^{2+\lceil\delta\rceil}\mathrm{d}s = O(1)$; and (c) $\tilde{\sigma}^2_{\boldsymbol{\delta}}(\boldsymbol{\theta}^*) \geq c_u$ for some constant $c_u > 0$, then under conditions C1-C3, C4'-C5' and N1-N2, we have that, as $L \to \infty$ and $m|D_n| \to \infty$,*

$$\frac{\sqrt{m|D_n|}\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)}{\tilde{\sigma}_{\boldsymbol{\delta}}(\boldsymbol{\theta}^*)} \xrightarrow{\mathcal{D}} N(0,1). \quad \text{(A.51)}$$

Proof. Following the exact same arguments in the proof of finding $mJ_{m,n}\text{Var}\left[H'_{m,n}(0)\right]$ in Lemma A.7, we have shown that

$$\text{Var}\left[\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)\right] = O(m^{-1}|D_n|^{-1}) \Rightarrow \tilde{\sigma}^2_{\boldsymbol{\delta}}(\boldsymbol{\theta}^*) = O(1).$$

To study the asymptotic normality of $\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)$, we define two random variables such that $\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*) = Z_1 - Z_2(\boldsymbol{\theta}^*)$ as follows

$$Z_1 = \frac{1}{m}\sum_{i=1}^{m}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|), \tag{A.52}$$

$$Z_2(\boldsymbol{\theta}^*) = \frac{1}{m(m-1)}\sum\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} w_R(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|). \tag{A.53}$$

By definition of $\boldsymbol{\theta}^*$ in (A.35), we have that

$$\mathbb{E}Z_1 = \mathbb{E}Z_2(\boldsymbol{\theta}^*) = \int_{D_n^2}\lambda(\mathbf{u})\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)g(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}. \tag{A.54}$$

We shall divide our discussions into two case scenarios: (1) $m\to\infty$ and (2) m is fixed.

Case I: when $m\to\infty$. In this case, the normality of Z_1 is easy to show since it is an average of independent and identically distributed random variables. The normality of $Z_2(\boldsymbol{\theta}^*)$ is less straightforward since it has a structure similar to a U-statistic, because

$$Z_2(\boldsymbol{\theta}^*) = \frac{1}{\binom{m}{2}}\sum_{i\neq j=1}^{m} Z_{2,i,j}(\boldsymbol{\theta}^*)$$

where

$$Z_{2,i,j}(\boldsymbol{\theta}^*) = \frac{1}{2}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j} w_R(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|).$$

To resolve issue, we define the following approximation

$$\tilde{Z}_2(\boldsymbol{\theta}^*) = \frac{2}{m}\sum_{i=1}^{m}\tilde{Z}_{2,i}(\boldsymbol{\theta}^*) - \mathbb{E}Z_2(\boldsymbol{\theta}^*), \tag{A.55}$$

where

$$\tilde{Z}_{2,i}(\boldsymbol{\theta}^*) = \sum_{\mathbf{u}\in X_i}\int_{D_n}\lambda(\mathbf{v})w_R(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\mathrm{d}\mathbf{v}.$$

It is trivial to see that $\mathbb{E}Z_2(\boldsymbol{\theta}^*) = \mathbb{E}\tilde{Z}_2(\boldsymbol{\theta}^*)$ by definition. Following similar arguments as those in the proof of finding $mJ_{m,n}\text{Var}\left[H'_{m,n}(0)\right]$ in Lemma A.2 and Lemma A.7, some tedious algebra gives that

$$
\begin{aligned}
& m\text{Var}\left[\tilde{Z}_2(\boldsymbol{\theta}^*) - Z_2(\boldsymbol{\theta}^*)\right] \\
&= \frac{2}{m-1}\int_{D_n^2} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)w_R^2(\|\mathbf{u}_1-\mathbf{v}_1\|)\tilde{g}_L^2(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)g(\|\mathbf{u}_1-\mathbf{v}_1\|)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]^2\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1 \\
&\quad + \frac{4}{m-1}\int_{D_n^3} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_1-\mathbf{u}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*) \\
&\qquad \times \tilde{g}_L(\|\mathbf{u}_1-\mathbf{u}_2\|;\boldsymbol{\theta}^*)\left[g(\|\mathbf{v}_1-\mathbf{u}_2\|)-1\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{u}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2 \\
&\quad + \frac{2}{m-1}\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)\tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*) \\
&\qquad \times g(\|\mathbf{u}_1-\mathbf{u}_2\|)\left[g(\|\mathbf{v}_1-\mathbf{v}_2\|)-1\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_2 \\
&\quad - \frac{2}{m-1}\int_{D_n^4} \lambda(\mathbf{u}_1)\lambda(\mathbf{v}_1)\lambda(\mathbf{u}_2)\lambda(\mathbf{v}_2)w_R(\|\mathbf{u}_1-\mathbf{v}_1\|)w_R(\|\mathbf{u}_2-\mathbf{v}_2\|)\left[g(\|\mathbf{u}_1-\mathbf{u}_2\|)-1\right] \\
&\qquad \times \tilde{g}_L(\|\mathbf{u}_1-\mathbf{v}_1\|;\boldsymbol{\theta}^*)\tilde{g}_L(\|\mathbf{u}_2-\mathbf{v}_2\|;\boldsymbol{\theta}^*)\left[\boldsymbol{\phi}_L(\|\mathbf{u}_1-\mathbf{v}_1\|)^T\boldsymbol{\delta}_L\right]\left[\boldsymbol{\phi}_L(\|\mathbf{u}_2-\mathbf{v}_2\|)^T\boldsymbol{\delta}_L\right]\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{u}_2\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2 \\
&= O(m^{-1}|D_n|^{-1})\int_0^R w_o^2(s)\left[\boldsymbol{\phi}_L(s)^T\boldsymbol{\delta}_L\right]^2 s^{d-1}\mathrm{d}s + O(m^{-1}|D_n|^{-1})\left\{\int_0^R w_o(s)\left[\boldsymbol{\phi}_L(s)^T\boldsymbol{\delta}_L\right]s^{d-1}\mathrm{d}s\right\}^2 \\
&= O(m^{-1}|D_n|^{-1}).
\end{aligned}
$$

Therefore, as $m\to\infty$, we have that

$$\sqrt{m|D_n|}\left[\tilde{Z}_2(\boldsymbol{\theta}^*) - Z_2(\boldsymbol{\theta}^*)\right] = O_p(m^{-1}) = o_p(1),$$

and hence

$$\sqrt{m|D_n|}\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*) = \sqrt{m|D_n|}\left[Z_1 - \tilde{Z}_2(\boldsymbol{\theta}^*)\right] + o_p(1).$$

Since $m|D_n|\text{Var}\left[\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)\right] \geq c_u$ for some constant $c_u > 0$, it suffices to show the asymptotic normality of

$$\sqrt{m|D_n|}\left[Z_1 - \tilde{Z}_2(\boldsymbol{\theta}^*)\right] = \frac{\sqrt{m|D_n|}}{m}\sum_{i=1}^m Y_i,$$

where Y_i's are i.i.d. random variables of the form as follows

$$Y_i = \sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|) - 2\tilde{Z}_{2,i}(\boldsymbol{\theta}^*) + \mathbb{E}Z_2(\boldsymbol{\theta}^*).$$

Note that $\sqrt{|D_n|}Y_i$'s are i.i.d random variables with a bounded variance (straightforward to show), (A.51) immediately follows from the standard central limit theorem as $m \to \infty$.

Case II: when m is fixed. In this case, condition $m|D_n| \to \infty$ requires that $|D_n| \to \infty$. In other words, we need to consider the case where the observation window of the point processes is expanding. Define a partition of $\mathbb{R}^d = \cup_{\mathbf{t}\in\mathbb{Z}^d}\Delta(\mathbf{t})$, where $\Delta(\mathbf{t}) = \prod_{k=1}^d (s(t_k - 1/2), s(t_k + 1/2)]$ with $s > 0$ as the length of the interval. Note that by this definition, $\Delta(\mathbf{t}_1) \cap \Delta(\mathbf{t}_2) = \emptyset$ if $\mathbf{t}_1 \neq \mathbf{t}_2 \in \mathbb{Z}$. Define random variables

$$Y_{1,n}(t) = \frac{|D_n|}{m}\sum_{i=1}^m \sum_{\mathbf{u}\in X_i\cap\Delta(t),\mathbf{v}\in X_i}\sum^{\neq} w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|),$$

$$Y_{2,n}(t) = \frac{|D_n|}{m(m-1)}\sum\sum_{i\neq j} \sum\sum_{\mathbf{u}\in X_i\cap\Delta(t),\mathbf{v}\in X_j} w_R(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|).$$

Then by definition, we have that

$$Z_1 = \frac{1}{|D_n|}\sum_{t\in\mathcal{T}_n} Y_{1,n}(\mathbf{t}), \quad Z_2(\boldsymbol{\theta}^*) = \frac{1}{|D_n|}\sum_{t\in\mathcal{T}_n} Y_{2,n}(\mathbf{t}),$$

where $\mathcal{T}_n = \{\mathbf{t}\in\mathbb{Z}^d : \Delta(\mathbf{t})\cap D_n \neq \emptyset\}$.

A simple application of the Jensen's inequality gives that $(m^{-1}\sum_{i=1}^m |x_i\|)^{2+\lceil\delta'\rceil} \leq m^{-1}\sum_{i=1}^m |x_i|^{2+\lceil\delta'\rceil}$ (note that $f(x) = |x|^{2+\lceil\delta'\rceil}$ is convex)

$$\begin{aligned}
\mathbb{E}\,|Y_{1,n}(t)|^{2+\lceil\delta'\rceil} &\leq \frac{1}{m}\sum_{i=1}^m \mathbb{E}\left|\sum_{\mathbf{u}\in X_i\cap\Delta(t),\mathbf{v}\in X_i}\sum^{\neq} |D_n| w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\right|^{2+\lceil\delta'\rceil} \\
&= \mathbb{E}\left|\sum_{\mathbf{u}\in X_1\cap\Delta(t),\mathbf{v}\in X_1}\sum^{\neq} |D_n| w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\right|^{2+\lceil\delta'\rceil} \\
&\leq \mathbb{E}\left\{\sum_{\mathbf{u}\in X_1\cap\Delta(t),\mathbf{v}\in X_1}\sum^{\neq} |D_n| w_R(\|\mathbf{u}-\mathbf{v}\|)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)|\right\}^{2+\lceil\delta'\rceil} \\
&= O(1)\mathbb{E}\left\{\sum_{\mathbf{u}\in X_1\cap\Delta(t),\mathbf{v}\in X_1}\sum^{\neq} I(\|\mathbf{u}-\mathbf{v}| < R) w_o(\|\mathbf{u}-\mathbf{v}\|)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)|\right\}^{2+\lceil\delta'\rceil},
\end{aligned}$$

where the last expectation is essentially bounded by sums of integrals involving $w_o^k(s)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(s)|^k$, $k \leq 2+\lceil\delta'\rceil$, $\lambda(\mathbf{u})$, $g(s)$, $g^{(k)}(\mathbf{u}_1,\ldots,\mathbf{u}_k)$, $k = 3,\ldots,2(2+\lceil\delta'\rceil)$. All terms are bounded under conditions C1-C3 and condition N2' except the first batch, hence we only need to consider upper bounds of integrals of the form

$$\int_0^R w_o^k(s)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(s)|^k \mathrm{d}s, \quad k \le 2+\lceil\delta'\rceil.$$

For any $k < 2+\lceil\delta'\rceil$, by the Höder's inequality with $p = (2+\lceil\delta'\rceil)/k, q = 1/[1-k/(2+\lceil\delta'\rceil)]$ such that $1/p+1/q = 1$, we have that

$$\begin{aligned}\int_0^R w_o^k(s)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(s)|^k \mathrm{d}s &= \int_0^R [w_o(s)]^{k-1+1/q}\left\{\left[w_o(s)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(s)|\right]^{2+\lceil\delta'\rceil}\right\}^{1/p}\mathrm{d}s \\ &\le \left\{\int_0^R \left[w_o(s)|\boldsymbol{\delta}_L^T\boldsymbol{\phi}_L(s)|\right]^{2+\lceil\delta'\rceil}\mathrm{d}s\right\}^{1/p}\left\{\int_0^R w_o^{q(k-1)+1}(s)\mathrm{d}s\right\}^{1/q} \\ &= O(1),\end{aligned}$$

where the last equality follows from the condition for $\boldsymbol{\delta}_L$. Therefore, we have that there exists a constant C_1 such that

$$\mathbb{E}\left|Y_{1,n}(\mathbf{t})\right|^{2+\lceil\delta'\rceil} < C_1.$$

Similar arguments also yield that for some constant $C_2 > 0$

$$\mathbb{E}\left|Y_{2,n}(\mathbf{t})\right|^{2+\lceil\delta'\rceil} < C_2.$$

Then by the Minkowski inequality, we have that

$$\begin{aligned}\mathbb{E}\left|Y_{1,n}(\mathbf{t})-Y_{2,n}(\mathbf{t})\right|^{2+\lceil\delta'\rceil} &\le \left\{\left[\mathbb{E}\left|Y_{1,n}(\mathbf{t})\right|^{2+\lceil\delta'\rceil}\right]^{1/(2+\lceil\delta'\rceil)} + \left[\mathbb{E}\left|Y_{2,n}(\mathbf{t})\right|^{2+\lceil\delta'\rceil}\right]^{1/(2+\lceil\delta'\rceil)}\right\}^{2+\lceil\delta'\rceil} \\ &< 2^{2+\lceil\delta'\rceil}\max\{C_1, C_2\},\end{aligned}$$

which further gives that

$$\sup_{n\ge1}\sup_{t\in\mathcal{T}_n}\mathbb{E}\left|Y_{1,n}(\mathbf{t})-Y_{2,n}(\mathbf{t})\right|^{2+\delta} \le \left(\sup_{n\ge1}\sup_{t\in\mathcal{T}_n}\mathbb{E}\left|Y_{1,n}(\mathbf{t})-Y_{2,n}(\mathbf{t})\right|^{2+\lceil\delta\rceil}\right)^{\frac{2+\delta}{2+\lceil\delta\rceil}} < \infty. \quad \text{(A.56)}$$

Note that the total number of disjoint partitions $\Delta(\mathbf{t})\cap D_n \ne \emptyset$ is of the order $|D_n|$, hence we can check that,

$$(|D_n|)^{-1}\mathrm{Var}\left\{\sum_{t\in\mathcal{T}_n}[Y_{1,n}(t)-Y_{2,n}(t)]\right\} = (|D_n|)^{-1}\mathrm{Var}\left(|D_n|\boldsymbol{\delta}_L^T\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)\right) = m^{-1}\tilde{\sigma}_{\boldsymbol{\delta}}^2(\boldsymbol{\theta}^*) \ge c_u/m.$$

Therefore, using conditions $N1'(b)$ and $N2'$, together with inequality (A.56), it follows from Theorem 1 of Biscio and Waagepetersen (2019) that as $|D_n| \to \infty$,

$$\left\{ \text{Var}\left[\sum_{t\in\mathcal{T}_n} Y_{1,n}(t) - \sum_{t\in\mathcal{T}_n} Y_{2,n}(t) \right] \right\}^{-1/2} \sum_{t\in\mathcal{T}_n} [Y_{1,n}(t) - Y_{2,n}(t)] \xrightarrow{\mathcal{D}} N(0,1),$$

which coincides with (A.51) by definition of $Y_{k,n}$'s, $k = 1, 2$. □

Lemma A.9. *Denote $\widehat{\boldsymbol{\theta}}$ as the solution to estimating equations $\tilde{\mathbf{U}}_L(\boldsymbol{\theta}) = \mathbf{0}$, then under conditions C1-C3 and C4'-C5', we have that as $L \to \infty$ and $L^{4\nu_0+2\nu_2}/m|D_n| \to 0$, for any $0 < r \le R$,*

$$\sqrt{m|D_n|}\boldsymbol{\phi}_L^T(r)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) = \sqrt{m|D_n|}\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*) + o_p(1)\left\|\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\right\|, \quad \text{(A.57)}$$

where $\boldsymbol{\theta}^*$ and $\mathbf{Q}_L$ are defined in (A.35) *and (14), respectively. Furthermore, under additional conditions N1-N3, we have that*

$$\frac{\sqrt{m|D_n|}\boldsymbol{\phi}_L^T(r)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*)}{\sigma_L(r;\boldsymbol{\theta}^*)} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{(A.58)}$$

where $\sigma_L^2(r;\boldsymbol{\theta}^*) = \boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\phi}_L(r)$ and $\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*) = \text{Var}\left[\sqrt{m|D_n|}\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*)\right]$.

Proof. Recall the definition

$$\tilde{\mathbf{U}}_L(\boldsymbol{\theta}) = \frac{1}{m}\sum_{i=1}^{m}\sum_{\mathbf{u},\mathbf{v}\in X_i}^{\neq} w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)$$
$$- \frac{1}{m(m-1)}\sum\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i,}\sum_{\mathbf{v}\in X_j} w_R(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}).$$

Using the first order Taylor expansion, we can show that

$$\tilde{\mathbf{U}}_L(\widehat{\boldsymbol{\theta}}) - \tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*) = -\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*), \quad \text{(A.59)}$$

where $\|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\| \le \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|$ and

$$\tilde{\mathbf{H}}_L(\boldsymbol{\theta}) = \sum\sum_{i\neq j=1}^{m}\sum_{\mathbf{u}\in X_i}\sum_{\mathbf{v}\in X_j}\frac{w_R(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)}\boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\phi}_L^T(\|\mathbf{u}-\mathbf{v}\|)\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}) \quad \text{(A.60)}$$

is a symmetric non-negative definite matrix. Observe that $\tilde{\mathbf{U}}_L(\widehat{\boldsymbol{\theta}}) = \mathbf{0}$, we can re-write expansion (A.59) as follows

$$\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*) = \left(\mathbf{Q}_L + \mathbf{Q}^{\Delta}\right)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*), \tag{A.61}$$

where $\mathbf{Q}_L$ is defined in (14) and $\mathbf{Q}^{\Delta} = \tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \mathbf{Q}_L$. From the above new expansion, we have that

$$\begin{aligned}\boldsymbol{\phi}_L^T(r)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) &= \boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\left[\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*) - \mathbf{Q}^{\Delta}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*)\right] \\ &\leq \boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*) + \left\|\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\right\|\sigma_{\max}\left[\mathbf{Q}^{\Delta}\right]\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|,\end{aligned} \tag{A.62}$$

where $\sigma_{\max}(\boldsymbol{A})$ stands for the largest singular value of the matrix $\boldsymbol{A}$. We have shown the order of $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|$ in Lemma (A.7), so it remains to quantify $\sigma_{\max}\left[\mathbf{Q}^{\Delta}\right]$. Note that we can further decompose $\mathbf{Q}^{\Delta}$ as follows

$$\mathbf{Q}^{\Delta} = \tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) + \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) - \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] + \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] - \mathbf{Q}_L.$$

By the property of the singular value, we readily have that

$$\sigma_{\max}\left[\mathbf{Q}^{\Delta}\right] \leq \sigma_{\max}\left[\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] + \sigma_{\max}\left\{\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) - \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right]\right\} + \sigma_{\max}\left\{\mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] - \mathbf{Q}_L\right\}, \tag{A.63}$$

which will be studied one by one.

By definition of $\tilde{\boldsymbol{\theta}}^*$ and Lemma A.7,

$$\sup_{0<r\leq R}\left|(\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*)^T\boldsymbol{\phi}_L(r)\right| \leq \|\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*\|\sup_{0<r\leq R}\|\boldsymbol{\phi}_L(r)\| = \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|O(L^{\nu_2}) = O_P\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right) = o_P(1).$$

Then, it is straightforward to see that

$$\begin{aligned}|\tilde{g}_L(r;\boldsymbol{\theta}^*) - \tilde{g}_L(r;\tilde{\boldsymbol{\theta}}^*)| &= \tilde{g}_L(r;\boldsymbol{\theta}^*)\left|1 - \exp\left[(\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*)^T\boldsymbol{\phi}_L(r)\right]\right| \\ &= \tilde{g}_L(r;\boldsymbol{\theta}^*)O(1)\left|(\tilde{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*)^T\boldsymbol{\phi}_L(r)\right| = \tilde{g}_L(r;\boldsymbol{\theta}^*)O_P\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right),\end{aligned}$$

which further implies that

$$\begin{aligned}
\eta_{\max}\left[\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] &= \sup_{\|\boldsymbol{\delta}\|=1} \boldsymbol{\delta}^T \left[\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] \boldsymbol{\delta} \\
&\le \sup_{\|\boldsymbol{\delta}\|=1} \sum\sum_{i\neq j=1}^{m} \sum_{\mathbf{u}\in X_i} \sum_{\mathbf{v}\in X_j} \frac{w_R(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)} [\boldsymbol{\delta}^T \boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)]^2 |\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|; \tilde{\boldsymbol{\theta}}^*) - \tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|; \boldsymbol{\theta}^*)| \\
&= O_P\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right) \sup_{\|\boldsymbol{\delta}\|=1} \sum\sum_{i\neq j=1}^{m} \sum_{\mathbf{u}\in X_i} \sum_{\mathbf{v}\in X_j} \frac{w_R(\|\mathbf{u}-\mathbf{v}\|)}{m(m-1)} [\boldsymbol{\delta}^T \boldsymbol{\phi}_L(\|\mathbf{u}-\mathbf{v}\|)]^2 \tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|; \boldsymbol{\theta}^*) \\
&= O_P\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right) \eta_{\max}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right].
\end{aligned}$$

Following exactly the same steps, we can also show that

$$-\eta_{\min}\left[\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] = \eta_{\max}\left[-\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) + \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] = \eta_{\max}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] O_P\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right),$$

which implies that

$$\sigma_{\max}\left[\tilde{\mathbf{H}}_L(\tilde{\boldsymbol{\theta}}^*) - \tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] = \eta_{\max}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] O_P\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}}\right), \tag{A.64}$$

where the convergence is entry-wise.

The next step is to quantify the magnitude of $\sigma_{\max}\left\{\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) - \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right]\right\}$. Using the standard random matrix theory, it suffices to consider the variability of $\boldsymbol{\delta}^T\left\{\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) - \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right]\right\}\boldsymbol{\delta}$ for any $\boldsymbol{\delta}\in\mathbb{R}^L$ with $\|\boldsymbol{\delta}\| = 1$. Following steps as those in the proof of Lemma A.7 about $\text{Var}\left[H''_{m,n}(t_z)\right]$, we immediately have that

$$\sup_{\|\boldsymbol{\delta}\|=1}\left|\boldsymbol{\delta}^T\left\{\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) - \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right]\right\}\boldsymbol{\delta}\right| = O_p\left(\frac{L^{2\nu_2}}{m^2|D_n|}\right) + O\left(\frac{1}{m|D_n|}\right),$$

hence that

$$\sigma_{\max}\left\{\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*) - \mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right]\right\} = O_p\left(\frac{L^{2\nu_2}}{m^2|D_n|}\right) + O\left(\frac{1}{m|D_n|}\right). \tag{A.65}$$

Next, we proceed to bound the largest singular value of $\mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] - \mathbf{Q}_L$. For any $\boldsymbol{\delta}\in\mathbb{R}^L$

with $\|\boldsymbol{\delta}\| = 1$,

$$\begin{aligned}
&\boldsymbol{\delta}^T \left\{\mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] - \mathbf{Q}_L\right\} \boldsymbol{\delta} \\
&= \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v}) w_R(\|\mathbf{u}-\mathbf{v}\|) \left[\boldsymbol{\phi}_L^T(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}\right]^2 \left[\tilde{g}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}^*) - g(\|\mathbf{u}-\mathbf{v}\|)\right] \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&= \int_{D_n^2} \lambda(\mathbf{u})\lambda(\mathbf{v}) w_R(\|\mathbf{u}-\mathbf{v}\|) \left[\boldsymbol{\phi}_L^T(\|\mathbf{u}-\mathbf{v}\|)\boldsymbol{\delta}\right]^2 g(\|\mathbf{u}-\mathbf{v}\|) \\
&\qquad\qquad \times \left\{1 - \exp\left[(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)^T \boldsymbol{\phi}(\|\mathbf{u}-\mathbf{v}\|) - \tilde{\zeta}_L(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}_0)\right]\right\} \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v} \\
&= O(1) \int_0^R w_o(s) \left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}\right]^2 \left[|(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)^T\boldsymbol{\phi}(s)| + |\tilde{\zeta}_L(s;\boldsymbol{\theta}_0)|\right] s^{d-1}\mathrm{d}s \\
&= O(1)\|\boldsymbol{\theta}^* - \boldsymbol{\theta}_0\| \sup_{0<r\le R} \|\boldsymbol{\phi}(r)\| \int_0^R w_o(s) \left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}\right]^2 \mathrm{d}s \\
&\quad + O(1)\sqrt{\int_0^R w_o(s)\tilde{\zeta}_L^2(\|\mathbf{u}-\mathbf{v}\|;\boldsymbol{\theta}_0)\mathrm{d}s} \times \sqrt{\int_0^R w_o(s) \left[\boldsymbol{\phi}_L^T(s)\boldsymbol{\delta}\right]^2 \mathrm{d}s} \\
&= O(L^{\nu_0+\nu_2-\nu_1}),
\end{aligned}$$

where the last equality follows from condition C4 and Lemma A.6. This further gives that

$$\sigma_{\max}\left\{\mathbb{E}\left[\tilde{\mathbf{H}}_L(\boldsymbol{\theta}^*)\right] - \mathbf{Q}_L\right\} = O(L^{\nu_0+\nu_2-\nu_1}). \tag{A.66}$$

Combining equations (A.63)-(A.65), we have that where $\sigma_{\max}\left(\mathbf{Q}^{\Delta}\right) = O_p\left(\frac{L^{\nu_0+\nu_2}}{\sqrt{m|D_n|}} + L^{\nu_0+\nu_2-\nu_1}\right)$. In addition, we have shown in Lemma A.7 that $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\| = O_p\left(\frac{L^{\nu_0}}{\sqrt{m|D_n|}}\right)$. Plugging these two equations back to (A.62), we have that

$$\boldsymbol{\phi}_L^T(r)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) = \boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*) + O_p\left(\frac{L^{2\nu_0+\nu_2}}{m|D_n|} + \frac{L^{\nu_0+\nu_2-\nu_1}}{\sqrt{m|D_n|}}\right)\left\|\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\right\|,$$

which gives (A.57), recall that $\nu_2+2\nu_0 < \nu_1$ in condition C5' and the condition $L^{4\nu_0+2\nu_2}/m|D_n| \to 0$.

To show (A.58), define vector $\boldsymbol{\ell}(r) = \left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\phi}_L^T(r)$ and its standardized version $\boldsymbol{\ell}_0(r) = \|\boldsymbol{\ell}(r)\|^{-1}\boldsymbol{\ell}(r)$ as in condition N3. Then applying Lemma A.8 to $\boldsymbol{\ell}_0^T(r)\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)$, under condition N3, we have that

$$\frac{\sqrt{m|D_n|}\boldsymbol{\ell}_0^T(r)\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)}{\sqrt{\boldsymbol{\ell}_0^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}_0(r)}} = \frac{\sqrt{m|D_n|}\boldsymbol{\ell}^T(r)\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)}{\sqrt{\boldsymbol{\ell}^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}(r)}} \xrightarrow{\mathcal{D}} N(0,1),$$

where $\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*) = \text{Var}\left[\sqrt{m|D_n|}\tilde{\mathbf{U}}(\boldsymbol{\theta}^*)\right]$. Then using (A.57), we have that

$$\begin{aligned}\frac{\sqrt{m|D_n|}\boldsymbol{\phi}_L^T(r)(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)}{\sqrt{\boldsymbol{\ell}^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}(r)}} &= \frac{\sqrt{m|D_n|}\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*)}{\sqrt{\boldsymbol{\ell}^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}(r)}} + o_p(1)\frac{\|\boldsymbol{\ell}(r)\|}{\sqrt{\boldsymbol{\ell}^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}(r)}}\\ &= \frac{\sqrt{m|D_n|}\boldsymbol{\ell}^T(r)\tilde{\mathbf{U}}_L(\boldsymbol{\theta}^*)}{\sqrt{\boldsymbol{\ell}^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}(r)}} + o_p(1)\frac{1}{\sqrt{\boldsymbol{\ell}_0^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}_0(r)}} \xrightarrow{\mathcal{D}} N(0,1),\end{aligned}$$

where the last equality follows from condition N3(a), which requires that $\sqrt{\boldsymbol{\ell}_0^T(r)\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\boldsymbol{\ell}_0(r)} \geq c_u$. The proof is complete. □

Proof of Theorem 2. Recall the definition $\tilde{g}_L(r;\boldsymbol{\theta}) = \exp\left[\boldsymbol{\theta}^T\boldsymbol{\phi}_L(r)\right]$ and $\hat{g}_L(r) = \tilde{g}_L(r;\widehat{\boldsymbol{\theta}})$, then applying the delta method to the asymptotic distribution of $\sqrt{m|D_n|}\boldsymbol{\phi}_L^T(r)(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)$ from Lemma A.9, we have that

$$\frac{\sqrt{m|D_n|}\left[\tilde{g}_L(r;\widehat{\boldsymbol{\theta}}) - \tilde{g}_L(r;\boldsymbol{\theta}^*)\right]}{\tilde{g}_L(r;\boldsymbol{\theta}^*)\sqrt{\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\phi}_L(r)}} \xrightarrow{\mathcal{D}} N(0,1).$$

By equation (A.37) in Lemma A.6, we have that $\sup_{0<r<R}|g(r) - \tilde{g}_L(r;\boldsymbol{\theta}^*)| = O(L^{-\nu_1+\tau_1} + L^{\nu_0-\nu_1+\nu_2}) = o(1)$, it readily follows that

$$\begin{aligned}\frac{\sqrt{m|D_n|}\left[\tilde{g}_L(r;\widehat{\boldsymbol{\theta}}) - \tilde{g}_L(r;\boldsymbol{\theta}^*)\right]}{\tilde{g}_L(r;\boldsymbol{\theta}^*)\sqrt{\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\phi}_L(r)}} &= \frac{\sqrt{m|D_n|}\left[\tilde{g}_L(r;\widehat{\boldsymbol{\theta}}) - g(r) + g(r) - \tilde{g}_L(r;\boldsymbol{\theta}^*)\right]}{\tilde{g}_L(r;\boldsymbol{\theta}^*)\sqrt{\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\phi}_L(r)}}\\ &= \frac{\sqrt{m|D_n|}\left[\tilde{g}_L(r;\widehat{\boldsymbol{\theta}}) - g(r) + O(L^{-\nu_1+\tau_1} + L^{\nu_0-\nu_1+\nu_2})\right]}{g(r)\sqrt{\boldsymbol{\phi}_L^T(r)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\Sigma}_U(\boldsymbol{\theta}^*)\left(\mathbf{Q}_L\right)^{-1}\boldsymbol{\phi}_L(r)}} + o_P(1) \xrightarrow{\mathcal{D}} N(0,1),\end{aligned}$$

which completes the proof. □

References

Biscio, C. A. N. and Waagepetersen, R. (2019), "A general central limit theorem and a subsampling variance estimator for α-mixing point processes," *Scandinavian Journal of Statistics*, 46, 1168–1190.

Burkill, J. C. and Burkill, H. (2002), *A second course in mathematical analysis*, Cambridge University Press.

Loader, C. R. et al. (1996), "Local likelihood density estimation," *The Annals of Statistics*, 24, 1602–1618.