A note on asymptotic results for estimating functions

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The following results are adapted from unpublished lecture notes by Professor Jens L. Jensen, University of Aarhus. We consider a parametrized family of probability measures P_θ, $\theta \in \mathbb{R}^p$, and a sequence of estimating functions $u_n : \mathbb{R}^p \to \mathbb{R}^p$, $n \geq 1$. The distribution of $\{u_n(\theta)\}_{n\geq 1}$ is governed by $P = P_{\theta^*}$ where θ^* denotes the 'true' parameter value. Similarly, we write $u_n = u_n(\theta^*)$ and $j_n = j_n(\theta^*)$ where $j_n(\theta)$ is the derivative of $-u_n(\theta)$. For a matrix $A = [a_{ij}]$, $\|A\| = \max_{ij} |a_{ij}|$.

The conditions for consistency and asymptotic normality are that there exist sequences a_n and c_n such that

1. $c_n a_n^{-2} \to 0$.
2. $u_n(\theta) = 0$ has almost surely a unique solution $\hat{\theta}_n$ for each n.
3. $j_n/a_n^2 \to F$ in probability under P for a positive definite matrix F.
4. For all $c > 0$,
$$\sup_{\|\theta-\theta^*\| a_n^2/c_n \leq c} \|j_n(\theta) - j_n\|/a_n^2 = \gamma_{nc} \to 0$$
in probability under P.
5. The normalized score function u_n/c_n is asymptotically zero-mean normal with covariance matrix Σ.

Note that the use of different normalizing sequences a_n and c_n is not standard in the literature on asymptotics for estimating functions. The second condition is assumed for ease of exposition and can be relaxed in view of the third condition. The following theorem ensures $O_p(c_n/a_n^2)$ consistency of $|\hat{\theta}_n - \theta^*|$ and asymptotic normality.

Theorem 1. *Under the conditions stated above, for each $\epsilon > 0$, there exists a $c > 0$ such that*

$$P(\|\hat{\theta}_n - \theta^*\| a_n^2/c_n < c) > 1 - \epsilon$$

whenever n is sufficiently large. Moreover,

$$(\hat{\theta}_n - \theta^*) a_n^2/c_n \to N(0, F^{-1}\Sigma F^{-1}).$$

Proof. Let $B_{nc} = \{\theta | \|\theta - \theta^*\| a_n^2/c_n < c\}$ be a ball of radius cc_n/a_n^2 around θ^* and $\hat{\theta}_n = \theta^* + \hat{\phi}_n c_n/a_n^2$. Note that $\hat{\theta}_n$ is in B_{nc} if $u_n(\theta^* + \phi c_n/a_n^2)\phi^{\mathsf{T}}/c_n < 0$ for all ϕ with $\|\phi\| = c$ since this implies $u_n(\theta^* + \phi c_n/a_n^2) = 0$ for some $\|\phi\| < c$ as a consequence of the fix point theorem. Hence we need to show that there is a c such that

$$P(\sup_{\|\phi\|=c} u_n(\theta^* + \phi c_n/a_n^2)\phi^{\mathsf{T}}/c_n \geq 0) \leq \epsilon \tag{1}$$

for sufficiently large n. To this end we write

$$u_n(\theta^* + \phi c_n/a_n^2)\phi^{\mathsf{T}}/c_n = u_n\phi^{\mathsf{T}}/c_n - \phi \int_0^1 j_n(\theta(t))/a_n^2 \mathrm{d}t \phi^{\mathsf{T}}$$

where $\theta(t) = \theta^* + t\phi c_n/a_n^2$. Then

$$\begin{aligned}
&P(\sup_{\|\phi\|=c} u_n(\theta^* + \phi c_n/a_n^2)\phi^{\mathsf{T}}/c_n \geq 0) \leq \\
&P(\sup_{\|\phi\|=c} u_n\phi^{\mathsf{T}}/c_n \geq \inf_{\|\phi\|=c} \phi \int_0^1 j_n(\theta(t))/a_n^2 \mathrm{d}t\phi^{\mathsf{T}}) \leq \\
&P(|u_n/c_n| \geq c[\inf_{\|\phi\|=1} \phi F \phi^{\mathsf{T}} - p\epsilon_n - p\gamma_{nc}]) \leq \\
&P(|u_n/c_n| \geq \lambda c/2) + P(p\epsilon_n + p\gamma_{nc} > \lambda/2)
\end{aligned}$$

where $\epsilon_n = \|j_n/a_n^2 - F\|$ and λ is the smallest eigenvalue for F. The first term can be made arbitrarily small by picking a sufficiently large c and letting n tend to infinity. The second term converges to zero as n tends to infinity. Hence (1) follows.

The asymptotic normality follows by showing that

$$(\hat{\theta}_n - \theta^*) F a_n^2/c_n - u_n/c_n$$

converges to zero in probability. Letting $\theta(t) = \theta^* + t(\hat{\theta}_n - \theta^*)$,

$$\begin{aligned}
&u_n/c_n = (\hat{\theta}_n - \theta^*)(a_n^2/c_n) \int_0^1 j_n(\theta(t))/a_n^2 \mathrm{d}t = \\
&(\hat{\theta}_n - \theta^*) F a_n^2/c_n + (\hat{\theta}_n - \theta^*)(a_n^2/c_n) \left[\int_0^1 (j_n(\theta(t)) - j_n)/a_n^2 \mathrm{d}t + j_n/a_n^2 - F\right].
\end{aligned}$$

It follows that if $\|(\hat{\theta}_n - \theta^*)a_n^2/c_n\| \leq c$ then $\|(\hat{\theta}_n - \theta^*)Fc_n/a_n^2 - u_n/c_n\| \leq cp(\epsilon_n + \gamma_{cn})$. Hence,

$$P(\|(\hat{\theta}_n - \theta^*)F\frac{a_n^2}{c_n} - u_n/c_n\| > \delta) \leq P(cp(\epsilon_n + \gamma_{cn}) > \delta) + P(\|(\hat{\theta}_n - \theta^*)\frac{a_n^2}{c_n}\| > c).$$

Here the first term converges to zero and the second term can be made arbitrarily small by picking a sufficiently large c and letting n tend to ∞. $\square$