Optimal first order estimating equations

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March 18, 2012

joint work

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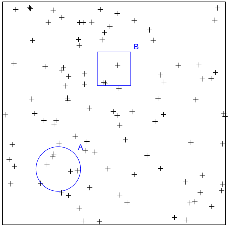
Mean and covariances of counts for spatial point process

Point process $\mathbf{X}$: random point pattern.

For A subset of the plane, count $N(A)$ is number of points in A.

$$\mathbb{E}N(A) = \int_A \rho(u)\mathrm{d}u$$



$\rho(\cdot)$: intensity function.

$$\mathbb{C}\mathrm{ov}[N(A), N(B)] = \int_{A\cap B} \rho(u)\mathrm{d}u + \int_A \int_B \rho(u)\rho(v)[g(u,v)-1]\mathrm{d}u\mathrm{d}v$$

$= $ Poisson proc. covariance $+$ extra term due to corr.

$g(u,v)$: pair correlation function

Campbell formulae

$$N(A) = \sum_{u \in \mathbf{X}} 1[u \in A]$$

$$N(A)N(B) = \sum_{u,v \in \mathbf{X}} 1[u \in A, u \in B]$$

Hence by moment formulae, for f function on $\mathbb{R}^2$ or $\mathbb{R}^2 \times \mathbb{R}^2$:

$$\mathbb{E} \sum_{u \in \mathbf{X}} f(u) = \int_{\mathbb{R}^2} f(u)\rho(u)\mathrm{d}u$$

$$\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} f(u,v) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(u,v)\rho(u)\rho(v)g(u,v)\mathrm{d}u\mathrm{d}v$$

Starting point for unbiased estimating functions !

Regression model for intensity function

Focus on estimation of parameter in regression model for intensity function.

E.g. log-linear model

$$\rho(u;\beta) = \exp[\beta Z(u)^{\mathsf{T}}]$$

where

$$Z(u) = (Z_1(u), \ldots, Z_p(u))$$

First-order estimating equations

Campbell $\Rightarrow$ unbiased *first-order* estimating function

$$u_f(\beta) = \sum_{u \in \mathbf{X} \cap W} f_\beta(u) - \int_W f_\beta(u)\rho_\beta(u)\mathrm{d}u$$

Choice

$$f_\beta(u) = \frac{\mathrm{d}}{\mathrm{d}\beta} \log \rho_\beta(u)$$

leads to composite likelihood/Poisson likelihood

$$\sum_{u \in \mathbf{X} \cap W} \frac{\rho'_\beta(u)}{\rho_\beta(u)} - \int_W \rho'_\beta(u)\mathrm{d}u$$

This is optimal choice for Poisson process (MLE) but what is optimal in the clustered case ?

Asymptotic results - first order estimating function

Let sensitivity

$$S_f = -\mathbb{E}\frac{\mathrm{d}}{\mathrm{d}\beta^\mathsf{T}} u_f(\beta)/|W|$$

and

$$\Sigma_f = \frac{\mathbb{V}\mathrm{ar}\, u_f(\beta)}{|W|}$$

Under appropriate mixing conditions,

$$\hat{\beta}_f \approx N(\beta, V_f/|W|)$$

where

$$V_f = S_f^{-1}\Sigma_f S_f^{-1}$$

Optimal first-order estimating equation

Optimal choice of f_β: smallest asymptotic variance

$$V_f = S_f^{-1} \Sigma_f S_f^{-1}$$

.

Optimal choice of f_β is solution of Fredholm equation

$$f_\beta(u) + \int_W t(u,v) f_\beta(v) \mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}\beta} \log \rho_\beta(u), \quad u \in W,$$

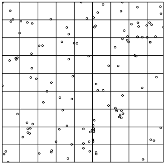
where integral equation kernel is

$$t(u,v) = \rho(v)[g(u,v) - 1]$$

Note: optimal f_β depends on pair correlation !

Numerical approximation and quasi-likelihood

Approximate solution of Fredholm equation using numerical quadrature: Riemann sum dividing W into cells C_i with representative points u_i.



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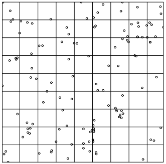

Resulting estimating function is *quasi-likelihood*

$$(N - \mu)V^{-1}D$$

based on

$$N = (N_1, \ldots, N_m), \quad N_i \text{ count of points in } C_i.$$

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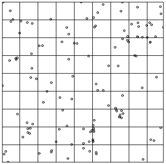

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μ mean of N:

$$\mu_i=\mathbb{E}N_i=\rho(u_i)|C_i| \text{ and } D=\left[\mathrm{d}\mu(u_i)/\mathrm{d}\beta_l\right]_{il}$$

V covariance of N:

$$V_{ij}=\mathbb{C}\mathrm{ov}[N_i,N_j]=\mu_i 1[i=j]+\mu_i\mu_j[g(u_i,u_j)-1]$$

Practical implementation: IGLS

Pair correlation function inside V estimated by e.g. minimum contrast.

Solve

$$(N - \mu(\beta)V(\beta)^{-1}D(\beta) = 0$$

using iterative generalized least squares:

$$(\beta^{(l+1)} - \beta^{(l)})D(\beta^{(l)})^\mathsf{T} V(\beta^{(l)})^{-1} D(\beta^{(l)}) = (N - \mu(\beta^{(l)}))V(\beta^{(l)})^{-1} D(\beta^{(l)})$$

One issue: use fine discretization (large m) $\Rightarrow$ V highdimensional matrix - e.g. V 10000×10000.

Use tapering and sparse matrix Cholesky from `Matrix` library in R.

Covariance matrix for $\hat{\beta}$:

$$S_{\text{taper}}^{-1} D^\mathsf{T} V_{\text{taper}}^{-1} V V_{\text{taper}}^{-1} D S_{\text{taper}}^{-1}, \quad S_{\text{taper}} = D^\mathsf{T} V_{\text{taper}}^{-1} D$$

Simulation study

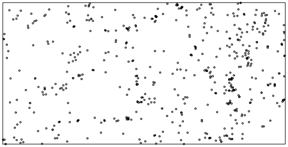
Consider variance of $\hat{\beta}$ obtained from either composite likelihood or quasi-likelihood.

Reduction in variance for quasi-likelihood relative to composite likelihood: 10% to 65%.

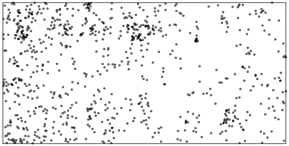
Large reductions when strong clustering and strong inhomogeneity.

Example: three tree species with different modes of seed dispersal:

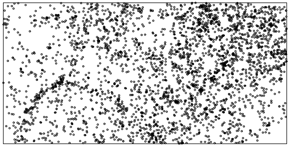
Acalypha Diversifolia

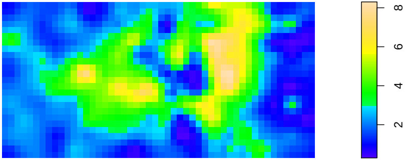


Loncocharpus Heptaphyllus



Capparis Frondosa

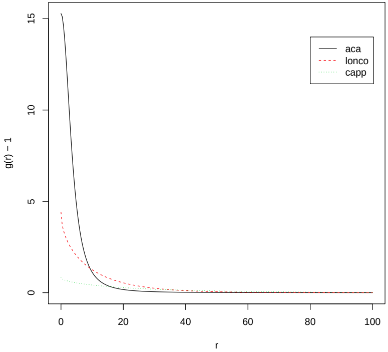




Potassium content in soil.

Covariates pH, elevation, gradient, potassium,...

Fitted pair correlation functions $g(\cdot)$



Acalypha: Cauchy.

Loncocharpus, Capparis: Matérn.

Results with composite likelihood and quasi-likelihood

species		$\widehat{\beta}$
Acalypha	CL	$-6.91 + 0.021\texttt{dem} + 0.0047\texttt{K}$ $(77.34^*, 9.77^*, 1.153^*) \times 10^{-3}$
	QL	$-6.90 + 0.016\texttt{dem} + 0.0047\texttt{K}$ $(77.09^*, 9.54, 1.133^*) \times 10^{-3}$
Loncocharpus	CL	$-6.49 - 0.021\texttt{Nmin} - 0.11\texttt{P} - 0.59\texttt{pH} - 0.11\texttt{twi}$ $(81.06^*, 7.45^*, 58.78, 282.89^*, 53.19^*) \times 10^{-3}$
	QL	$-6.49 - 0.023\texttt{Nmin} - 0.12\texttt{P} - 0.55\texttt{pH} - 0.084\texttt{twi}$ $(80.15^*, 6.95^*, 55.23^*, 266.10^*, 45.47) \times 10^{-3}$
Capparis	CL	$-5.07 + 0.028\texttt{dem} - 1.10\texttt{grad} + 0.0043\texttt{K}$ $(79.54^*, 9.98^*, 1200.36, 1.16^*) \times 10^{-3}$
	QL	$-5.10 + 0.019\texttt{dem} - 2.50\texttt{grad} + 0.0039\texttt{K}$ $(77.77^*, 8.86^*, 935.02^*, 1.02^*) \times 10^{-3}$

Estimated standard errors always smallest for QL. Regression parameters similar except for grad, Capparis.

Thanks for your attention