Estimating functions for inhomogeneous spatial point processes with incomplete covariate data

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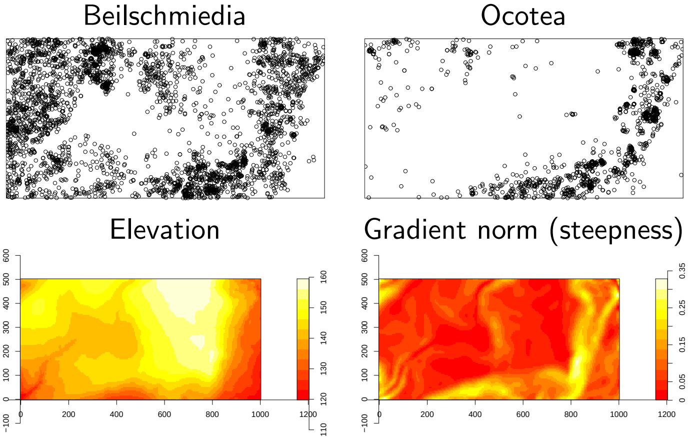
Data (Barro Colorado Island Forest Dynamics Plot)

Observation window: $S = [0, 1000] \times [0, 500] \text{m}^2$



Question: tree intensities related to elevation and gradient ?

Log-linear models for intensity function

$z(u) = \big(z_1(u), \ldots, z_p(u)\big)$ vector of covariates for each location u in observation window W.

E.g. $z(u) = (1, z_{\text{elev}}(u), z_{\text{grad}}(u))$ for rain forest example

Log-linear model for intensity function:

$$\lambda(u; \beta) = \exp(z(u)\beta^{\mathsf{T}})$$

Interpretation:

$$\lambda(u; \beta)\mathrm{d}N_u = P(\text{point occurs in neighbourhood } N_u \text{ around } u)$$

Poisson process case: log likelihood function and derivatives

$\mathbf{x}$ observation of $\mathbf{X}$ Poisson$(W, \lambda(\cdot;\beta))$.

Density wrt. unit rate Poisson process:

$$f(\mathbf{x};\beta) = \exp(|W| - \int_W \lambda(u;\beta)\mathrm{d}u) \prod_{u\in\mathbf{x}} \lambda(u;\beta)$$

log likelihood function:

$$l(\beta) = \sum_{u\in\mathbf{x}} z(u)\beta^\mathsf{T} - \int_W \lambda(u;\beta)\mathrm{d}u$$

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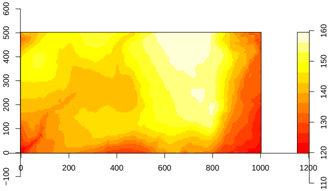
Score function and Fisher information:

$$u(\beta) = \sum_{u\in\mathbf{x}} z(u) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u \quad j(\beta) = \int_W z(u)^{\mathsf{T}} z(u)\lambda(u;\beta)\mathrm{d}u$$

$$\hat{\beta} \approx N(\beta, V) \quad V = j(\beta)^{-1}$$

Missing covariate data

Elevation covariate



interpolated from elevation observations on grid.

However, evaluating

$$u(\beta) = \sum_{u \in \mathbf{x}} z(u) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u$$

requires $z(u)$ *observed* for any $u \in W$!

Approximations of log likelihood I

Suppose $z(u)$ observed at finite set of locations $\mathbf{Q} \subset W$.

Rathbun (1996) approximate

$$\int_W z(u)\lambda(u;\beta)\mathrm{d}u \approx \int_W \widehat{z(u)\lambda(u;}\beta)\mathrm{d}u$$

where $\widehat{z(u)\lambda(u;}\beta)$ unbiased prediction of $z(u)\lambda(u;\beta)$, $u \in W$ given $z(u)$, $u \in \mathbf{Q}$.

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Riemann approximation:

$$\int_W z(u)\lambda(u;\beta)\mathrm{d}u \approx \sum_{u \in \mathbf{Q}} w(u)z(u)\lambda(u;\beta)$$

where $w(u)$ quadrature weight for $u \in \mathbf{Q}$.

Approximations of log likelihood II: spatstat

Approximation of score function used in R package spatstat (Baddeley and Turner)

$$u(\beta) \approx u^{\text{spat}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{Q}} w(u) z(u) \lambda(u; \beta)$$

but now $\mathbf{Q} = \mathbf{X} \cup \mathbf{D}$ *includes observed points* in addition to 'dummy' points $\mathbf{D}$.

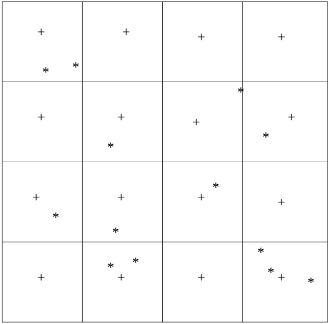
Note: $\mathbf{X}$ events of point pattern (supplied by 'nature')

$\mathbf{D}$ dummy points controlled by scientist.

Two types of weights: `grid` or `dirichlet`

Quadrature schemes in spatstat

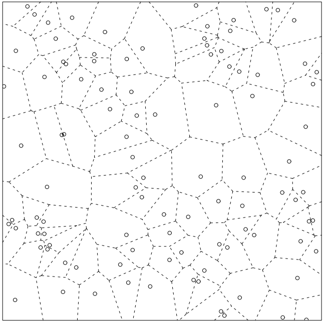
Grid



$$w(u) = \frac{|C_v|}{\#(\mathbf{X} \cap C_v) + 1}, \; u \in C_v$$

where $W = \cup_{v \in \mathbf{D}} C_v$

Dirichlet



$w(u)$ area of *Dirichlet cell* for u in Dirichlet tesselation generated by $\mathbf{Q}$.

`spatstat`: relation to generalized linear models and iterative weighted least squares

Estimating function

$$u^{\text{spat}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} w(u) z(u) \lambda(u; \beta)$$

formally equivalent to score function of weighted Poisson regression (log link):

$$u^{\text{spat}}(\beta) = \sum_{u \in \mathbf{X} \cup \mathbf{D}} w(u) z(u) \big(y_u - \lambda(u; \beta) \big)$$

with weights $w(u)$ and 'observations' $y_u = 1[u \in \mathbf{X}]/w(u)$.

Hence implementation straightforward using e.g. `glm()` in `R` (Poisson family, log link).

Distribution of parameter estimates from approximate score functions

?

Problem: hard to obtain distribution of parameter estimates from approximate score functions

Monte Carlo approximation of integral

Consider M *random* uniform dummy points on W (*simple* random dummy points). Assume wlog $|W| = 1$.

Rathbun et al. (2006): Monte Carlo approx. of integral:

$$u^{rath}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{D}} \frac{z(u)\lambda(u;\beta)}{M}$$

CLT for Monte Carlo approximation:

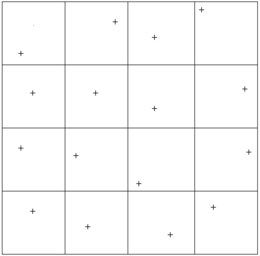
$$M^{1/2}\Big[\sum_{u \in \mathbf{D}_n} \frac{f(u)}{M} - \int_W f(u)\mathrm{d}u\Big] \xrightarrow{d} N(0, G_f)$$

where

$$G_f = \int_W f(u)^{\mathsf{T}} f(u)\mathrm{d}u - \frac{1}{|W|}\int_W f(u)^{\mathsf{T}}\mathrm{d}u \int_W f(u)\mathrm{d}u.$$

Stratified dummy points

Alternative: one uniformly sampled dummy point in each cell (stratified dummy points)



Suppose f continuously differentiable. Then CLT

$$M\left[\sum_{u\in\mathbf{D}_n}\frac{f(u)}{M}-\int_W f(u)\mathrm{d}u\right]\xrightarrow{d} N(0,G_f)$$

where M increasing number of dummy points and

$$G_f=\frac{1}{12}\int_W A_f(u)\mathrm{d}u \quad A_f(u_1,u_2)=\left[\frac{\partial f_i}{\partial u_1}\frac{\partial f_j}{\partial u_1}+\frac{\partial f_i}{\partial u_2}\frac{\partial f_j}{\partial u_2}\right]$$

(faster rate of convergence).

Distribution of parameter estimate

Consider increasing intensity asymptotics: intensities

$$\lambda_n(u;\beta) = n\lambda(u;\beta),\ \beta \in \mathbb{R}^p \quad \text{and} \quad M_n = n^k \rho$$

for observed $\mathbf{X}_n$ and dummy $\mathbf{D}_n$ ($k = 1$ (simple) or $1/2$ (strat.))
ρ: controls proportion of dummy points.

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$$u_n(\beta) + n[\int_W z(u)\lambda(u;\beta)\mathrm{d}u - \sum_{u\in\mathbf{D}_n} \frac{z(u)\lambda(u;\beta)}{n^k\rho}]$$

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Note $\mathbf{X}_n \sim \cup_{i=1}^n \mathbf{X}^i$ where $\mathbf{X}^i$ *iid* Poisson processes $\lambda(u;\beta) \Rightarrow$ CLT.

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Note $\mathbf{X}_n \sim \cup_{i=1}^n \mathbf{X}^i$ where $\mathbf{X}^i$ *iid* Poisson processes $\lambda(u;\beta) \Rightarrow$ CLT.
Hence,

$$n^{-1/2} u_n^{rath}(\beta) \xrightarrow{d} N(0, j(\beta) + G_k/\rho^{1/k}),$$

$$n^{1/2}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, V + VG_k V/\rho^{1/k}) \quad V = j(\beta)^{-1}$$

Monte Carlo versions of `spatstat` (Waagepetersen, 2007)

D point process of dummy points of intensity ρ. Monte Carlo version of `dirichlet`

$$u^{\text{dir}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} z(u) \lambda(u; \beta) \frac{1}{\lambda(u; \beta) + M}$$

(either simple or stratified dummy points)

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Monte Carlo version of `grid`: (stratified dummy points)

$$u^{\text{grid}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} z(u)\lambda(u;\beta)\frac{1}{M(\#(\mathbf{X} \cap C_u) + 1)}$$

Advantage: implementation using `glm()` just as for usual `spatstat`.

Asymptotic distribution of parameter estimates I

Grid version ($k = 1/2$)

$$u_n^{\text{grid}}(\beta) = \sum_{u \in \mathbf{X}_n} z(u) - \sum_{v \in \mathbf{D}_n} \frac{n}{n^{1/2}\rho} \frac{z(v)\lambda(v;\beta) + \sum_{u \in \mathbf{X} \cap C_{v,n}} z(u)\lambda(u;\beta)}{\#(\mathbf{X} \cap C_{v,n}) + 1}$$

$$\approx u_n^{rath}(\beta) = \sum_{u \in \mathbf{X}_n} z(u) - \sum_{u \in \mathbf{D}_n} \frac{z(u) n \lambda(u;\beta)}{n^{1/2}\rho}$$

Assuming continuously differentiable covariates

$$n^{-1/2} u_n^{\text{grid}}(\beta) \sim n^{-1/2} u_n^{\text{rath}}(\beta) \quad n \to \infty$$

and asymptotic covariance matrix becomes

$$V + V G_{1/2} V / \rho^2$$

Tends to MLE asymp. cov. V if $\rho \to \infty$.

Asymptotic distribution of parameter estimates II

Dirichlet estimating function:

$$u_n^{\mathrm{dir}}(\beta) = \sum_{u \in \mathbf{X}_n} z(u) - \sum_{u \in \mathbf{X}_n \cup \mathbf{D}_n} z(u) \frac{\lambda(u;\beta)}{\lambda(u;\beta) + n^{k-1}\rho}$$

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$$n^{-k+1/2} u_n^{\mathrm{dir}}(\beta) \xrightarrow{d} N(0, \rho^2 C_k + \rho^{2-1/k} H_k)$$

$$n^{-k} j_n^{\mathrm{dir}}(\beta) = -n^{-k} \frac{\mathrm{d}}{\mathrm{d}\beta} u_n^{\mathrm{dir}}(\beta) \xrightarrow{p} \rho F_k$$

Note: normalizing sequences $n^{-k+1/2}$ and n^{-k} depend on k.

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Hence

$$n^{1/2}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, F_k^{-1} C_k F_k^{-1} + F_k^{-1} H_k F_k^{-1}/\rho^{1/k})$$

Case $k = 1/2$: $F_k^{-1} C_k F^{-1}$ differs from V even when $\rho \to \infty$!

Numerical example: Poisson process

Case of Poisson process with covariate vector $(1, z_{\text{elev}})$ $\beta_{\text{elev}} = 0.1$.

Ratios of asymptotic standard errors for estm. funct. estimate $\hat{\beta}_{\text{elev}}$ relative to asymptotic standard error for MLE.

	simple					stratified			
q	0.25	1	10	100		0.25	1	10	100
u^{rath}	2.47	1.51	1.06	1.01	u^{grid}	1.08	1.01	1.00	1.00
u^{dir}	2.12	1.43	1.06	1.01	u^{dir}	1.56	1.53	1.53	1.53

$q = \#\mathbf{D}/\#\mathbf{X}$ proportion of dummy points.

u^{dir} better than Rathbuns Monte Carlo approximation u^{rath} but not useful in case of stratified dummy points.

Perspective

- methodology available for handling missing covariate data
- implementation straightforward
- also works for cluster point processes
- random sampling schemes required

References:

Waagepetersen, R. (2007). An estimating function approach to inference for inhomogeneous Neyman-Scott processes, *Biometrics*, **63**, 252-258.

Waagepetersen, R. (2007) Estimating functions for inhomogeneous spatial point processes with incomplete covariate data, submitted.