Statistical inference for inhomogeneous Cox processes

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Outline

Inhomogeneous clustered data sets

Estimating functions for inhomogeneous Cox processes

Maximum likelihood inference for thinned Cox processes

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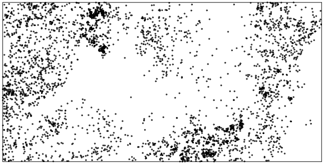
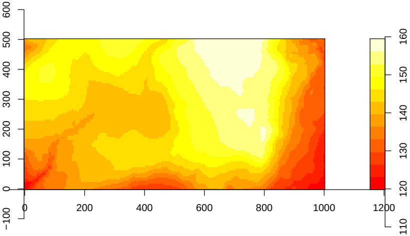
Tropical rain forests trees

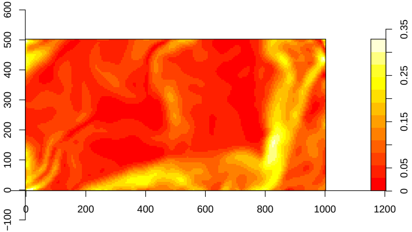
Beilschmiedia



- *observation window* $= 1000 \text{ m} \times 500 \text{ m}$
- seed dispersal $\Rightarrow$ *clustering*
- *covariates* $\Rightarrow$ *inhomogeneity*



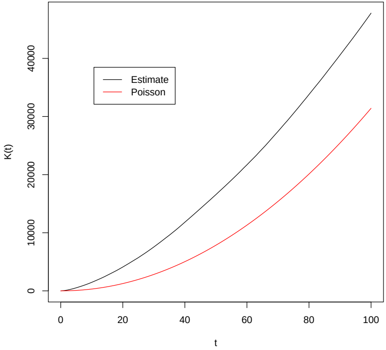
Altitude



Norm of altitude gradient (steepness)

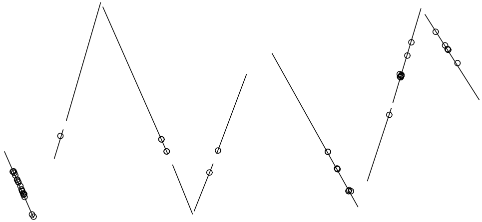
K-function (adjusted for inhomogeneity due to covariates)

Estimate of K adjusted for inhomogeneous intensity function $\rho(u)$:

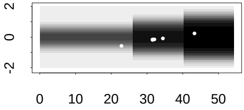


Poisson process not appropriate.

Whale positions



Close up:

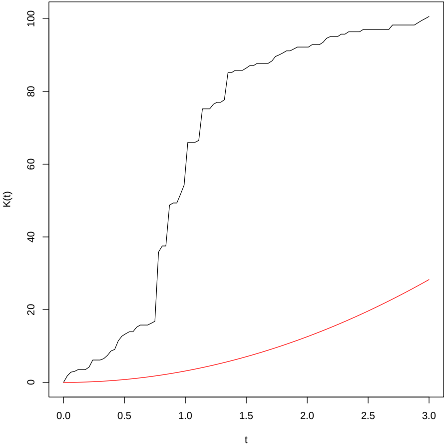


Observation window W = narrow strips around transect lines

Varying detection probability: inhomogeneity (thinning)

Variation in prey intensity: clustering

K-function (adjusted for inhomogeneity due to thinning)



Poisson process not appropriate.

Inhomogeneous clustered data sets

Estimating functions for inhomogeneous Cox processes

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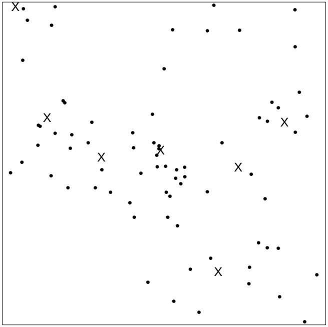
Cluster process (Thomas process)



Mothers (crosses) Poisson point process $\mathbf{\Phi}$ with intensity $\kappa > 0$.

Offspring $\mathbf{X} = \cup_{c \in \mathbf{\Phi}} \mathbf{X}_c$ distributed around mothers c according to bivariate Gaussian density f.

ω: standard deviation of Gaussian density

α: mean of Poisson number of offspring for each mother.

Random intensity function:

$$\Lambda(u) = \alpha \sum_{c \in \mathbf{\Phi}} f(u - c; \omega)$$

Inhomogeneous Cox process

$z_{1:p}(u) = (z_1(u), \ldots, z_p(u))$ vector of p nonconstant covariates.

$\beta_{1:p} = (\beta_1, \ldots \beta_p)$ regression parameter.

Random intensity function:

$$\Lambda(u) = \alpha \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) \sum_{c \in \mathbf{\Phi}} f(u - c; \omega)$$

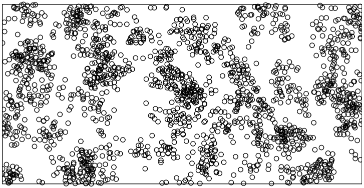
Rain forest example:

$$z_{1:2}(u) = (z_{\text{elev}}(u), z_{\text{grad}}(u))$$

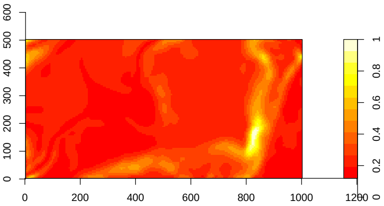
elevation/gradient covariate

Interpretation in terms of thinning

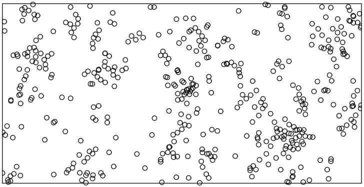
Homogeneous Cox process



Survival probabilities
$p(u) \propto \exp(z_{1:2}(u)\beta_{1:2}^{\mathsf{T}})$



After thinning (inhomogeneous Cox)



Parameter Estimation: regression parameters

Intensity function for inhomogeneous Cox:

$$\rho_\beta(u) = \kappa\alpha \exp(z(u)_{1:p}\beta_{1:p}^\mathsf{T}) = \exp(z(u)\beta^\mathsf{T})$$
$$z(u) = (1, z_{1:p}(u)) \quad \beta = (\log(\kappa\alpha), \beta_{1:p})$$

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Consider indicators $N_i = \mathbf{1}[\mathbf{X} \cap C_i \neq \emptyset]$ of occurrence of points in disjoint C_i ($W = \cup C_i$) where $P(N_i = 1) \approx \rho_\beta(u_i)\mathrm{d}C_i$, $u_i \in C_i$.

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Composite likelihood

$$\prod_{i=1}^{n} (\rho_\beta(u_i)\mathrm{d}C_i)^{N_i}(1-\rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i} \equiv \prod_{i=1}^{n} \rho_\beta(u_i)^{N_i}(1-\rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i}$$

Limit ($\mathrm{d}C_i \to 0$) of log composite likelihood

$$l(\beta) = \sum_{u \in \mathbf{X}\cap W} \log \rho_\beta(u) - \int_W \rho_\beta(u)\,\mathrm{d}u$$

Maximize using `spatstat` to obtain $\hat{\beta}$.

Asymptotic distribution of regression parameter estimates

Assume increasing mother intensity: $\kappa_n = n\tilde{\kappa} \to \infty$ and
$\mathbf{\Phi} = \cup_{i=1}^{n} \mathbf{\Phi}_i$, $\mathbf{\Phi}_i$ independent Poisson processes of intensity $\tilde{\kappa}$.

Score function asymptotically normal:

$$\frac{1}{\sqrt{n}} \frac{\mathrm{d}l(\beta)}{\mathrm{d}\log\alpha \mathrm{d}\beta_{1:p}} = \frac{1}{\sqrt{n}} \left(\sum_{u \in \mathbf{X} \cap W} z(u) - n\tilde{\kappa}\alpha \int_W z(u) \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) \mathrm{d}u \right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big[\sum_{c \in \mathbf{\Phi}_i} \sum_{u \in \mathbf{X}_c \cap W} z(u) - \tilde{\kappa}\alpha \int_W \exp(z_{1:p}(u)\beta_{1:p}^{\mathsf{T}}) \mathrm{d}u \Big] \approx N(0, V)$$

where $V = \mathbb{V}\mathrm{ar} \sum_{c \in \mathbf{\Phi}_i} \sum_{u \in \mathbf{X}_c \cap W} z(u)$

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By standard results for estimating functions (J observed information for Poisson likelihood):

$$\sqrt{\kappa_n} \big[(\log(\hat{\alpha}), \hat{\beta}_{1:p}) - (\log \alpha, \beta_{1:p}) \big] \approx N(0, J^{-1} V J^{-1})$$

Parameter Estimation: clustering parameters

Theoretical expression for (inhomogeneous) K-function:

$$K(t;\kappa,\omega) = \pi t^2 + \big(1 - \exp(-t^2/(2\omega)^2)\big)/\kappa.$$

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Semi-parametric estimate

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u - v\| \leq t]}{\rho_{\hat{\beta}}(u)\rho_{\hat{\beta}}(v)|W \cap W_{u-v}|}$$

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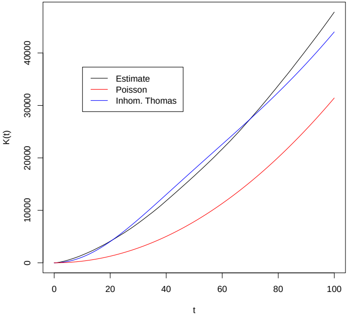
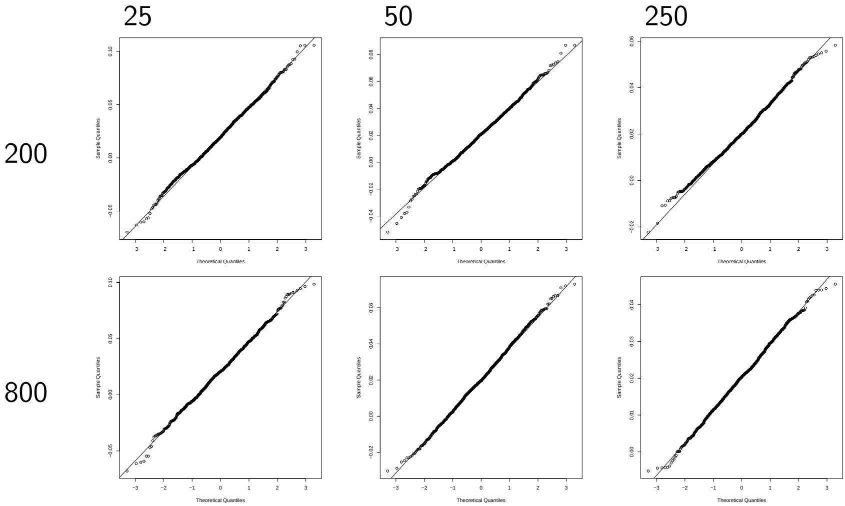
Estimate κ and ω by minimizing contrast

$$\int_0^{100} \left(K(t;\kappa,\omega)^{1/4} - \hat{K}(t)^{1/4}\right)^2 \mathrm{d}t$$



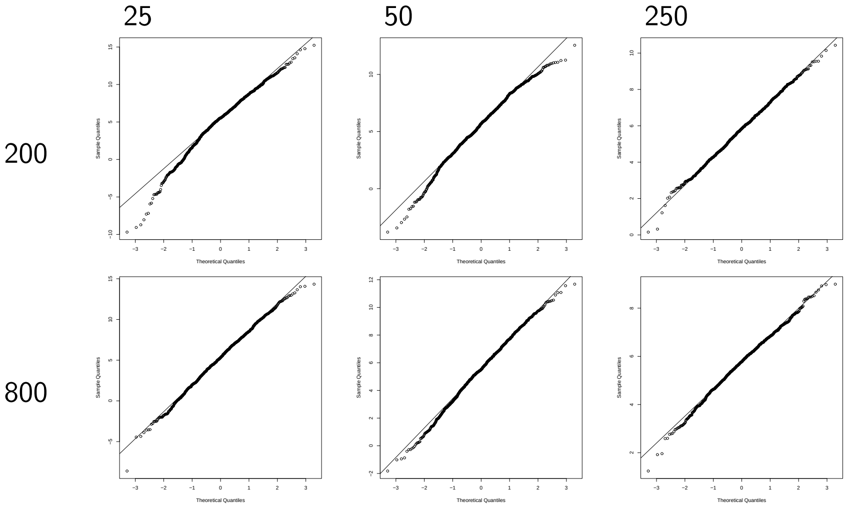
Simulation study

Quantile plots of $\hat{\beta}_{\text{elev}}$ (varying expected numbers 25, 50 and 250 of mothers and offspring, 200 or 800)



Simulation study II

Quantile plots of $\hat{\beta}_{\text{grad}}$ (varying expected numbers 25, 50 and 250 of mothers and offspring, 200 or 800)



Results for Beilschmiedia

Parameter estimates and confidence intervals (Poisson in red).

Elevation	Gradient	κ	α	ω
0.021 [-0.018,0.061]	5.842 [0.885,10.797]	8e-05	85.9	20.0
[0.017,0.026]	[5.340,6.342]			

Clustering: less information in data and wider confidence intervals than for Poisson process (independence).

Evidence of positive association between gradient and Beilschmiedia intensity.

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Shot-noise Cox process model for whales

Whales: stationary Cox process $\mathbf{Y}$ with random intensity function

$$\Lambda(u) = \sum_{(c,\gamma)\in\mathbf{\Phi}} \gamma k(u - c)$$

$\mathbf{\Phi}$ homogeneous marked Poisson process of marked cluster centres (c, γ) where $\gamma \sim \Gamma(\alpha, 1)$.

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$p(u)$ detection probability of observing whale at location u.

Observed whales: $\mathbf{X}$ thinning of all whales $\mathbf{Y}$ i.e. inhomogeneous Cox process with random intensity function

$$p(u)\Lambda(u)$$

Note: $\mathbf{X}_{\neg\mathrm{obs}} = \mathbf{Y} \setminus \mathbf{X}$ and $\mathbf{X}$ independent Poisson processes given $\mathbf{\Phi}$.

Parameters

Assume $k(\cdot)$ bivariate Gaussian density truncated to have bounded support.

Parameters:

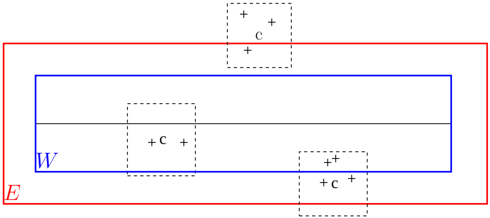
κ intensity of cluster centres c

$\alpha = \mathbb{E}\gamma$ (expected cluster size)

ω standard deviation of Gaussian density

Likelihood function for one transect

A transect:



W: support of $p(\cdot)$. E: $k(u-c)=0$ if $c \in \mathbb{R}^2 \setminus E$ and $u \in W$.

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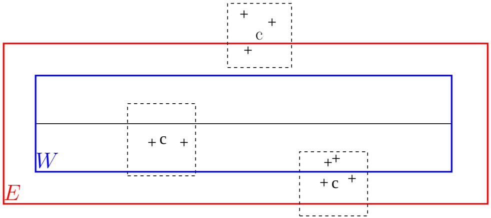

W: support of $p(\cdot)$. E: $k(u-c)=0$ if $c \in \mathbb{R}^2 \setminus E$ and $u \in W$.

Likelihood: $\theta = (\kappa, \alpha, \omega)$

1. $\mathbf{x}$ observed whales in W with conditional Poisson density

$$f(\mathbf{x}|\mathbf{\Phi};\omega) = \exp\Big(\int_W (1 - p(u)\Lambda(u))\mathrm{d}u\Big) \prod_{u \in \mathbf{x}} p(u)\Lambda(u)$$

2.

$$L(\theta) = \mathbb{E}_{(\kappa,\alpha)} f_\theta(\mathbf{x}|\mathbf{\Phi};\omega) = \mathbb{E}_{(\kappa,\alpha)} f(\mathbf{x}|\mathbf{\Phi} \cap E;\omega)$$

Derivatives of likelihood function

$\mathbf{\Phi}_E = \mathbf{\Phi} \cap E$ finite marked Poisson process with density

$$f(\phi; \kappa, \alpha) = \mathrm{e}^{|E|(1-\kappa)} \kappa^{n(\phi)} \prod_{(c,\gamma)\in\phi} \gamma^{\alpha-1} \exp(-\gamma)/\Gamma(\alpha)$$

Joint density of $\mathbf{X}$ and $\mathbf{\Phi}_E$:

$$f(\mathbf{x}, \phi; \kappa, \alpha, \omega) = f(\mathbf{x}|\phi; \omega) f(\phi; \kappa, \alpha)$$

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Let

$$V_\theta(\mathbf{X}, \mathbf{\Phi}_E) = \mathrm{d} \log f(\mathbf{X}, \mathbf{\Phi}_E; \theta)/\mathrm{d}\theta$$

Score function and observed information

$$u(\kappa, \alpha) = \frac{\mathrm{d} \log L(\theta)}{\mathrm{d}\theta} = \mathbb{E}_\theta[V_\theta(\mathbf{X}, \mathbf{\Phi}_E)|\mathbf{X} = \mathbf{x}] \quad \text{and}$$

$$j(\kappa, \alpha) = -\mathbb{E}_\theta[\frac{\mathrm{d} V_\theta(\mathbf{X}, \mathbf{\Phi}_E)}{\mathrm{d}\theta^\mathsf{T}}|\mathbf{X} = \mathbf{x}] - \mathbb{V}\mathrm{ar}_\theta[V_\theta(\mathbf{X}, \mathbf{\Phi}_E)|\mathbf{X} = \mathbf{x}]$$

Importance sampling

$\theta = (\kappa, \alpha, \omega)$

$\mathbf{\Phi}_0, \mathbf{\Phi}_1, \dots, \mathbf{\Phi}_{n-1}$ sample from $f(\phi|\mathbf{x};\theta_0) = f(\mathbf{x},\phi;\theta_0)/f(\mathbf{x};\theta_0)$ for fixed $\theta_0 = (\kappa_0, \alpha_0, \omega_0)$

$$\begin{aligned}
\mathbb{E}_\theta[k(\mathbf{\Phi})|\mathbf{X}=\mathbf{x}] &= \frac{f(\mathbf{x};\theta_0)}{f(\mathbf{x},\theta)}\mathbb{E}_{\theta_0}\left[k(\mathbf{\Phi})\frac{f(\mathbf{x},\mathbf{\Phi};\theta)}{f(\mathbf{x};\mathbf{\Phi};\theta_0)}|\mathbf{X}=\mathbf{x}\right] \\
&\approx \frac{f(\mathbf{x};\theta_0)}{f(\mathbf{x},\theta)}\frac{1}{n}\sum_{m=0}^{n-1} k(\mathbf{\Phi}_m)\frac{f(\mathbf{x},\mathbf{\Phi}_m;\theta)}{f(\mathbf{x};\mathbf{\Phi}_m;\theta_0)}
\end{aligned}$$

$$\frac{f(\mathbf{x};\theta)}{f(\mathbf{x},\theta_0)} = \frac{L(\theta)}{L(\theta_0)} \approx \frac{1}{n}\sum_{m=0}^{n-1}\frac{f(\mathbf{x},\mathbf{\Phi}_m;\theta)}{f(\mathbf{x};\mathbf{\Phi}_m;\theta_0)}$$

Hence Monte Carlo approximations of likelihood ratios, score, and observed information.

Markov chain Monte Carlo

Conditional density of $\boldsymbol{\Phi}_E$ given $\mathbf{X} = \mathbf{x}$:

$$f(\phi|\mathbf{x}) \propto f(\phi)f(\mathbf{x}|\phi) = f(\phi)\,\mathrm{e}^{-\int_W p(u)\Lambda(u|\phi)\mathrm{d}u} \prod_{u\in\mathbf{x}} p(u)\Lambda(u|\phi)$$

Computation of $\int_W p(u)\Lambda(u|\phi)\mathrm{d}u$ not straightforward.

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Demarginalisation impute $\mathbf{X}_{\neg\text{obs}} = (\mathbf{Y} \cap W) \setminus \mathbf{X}$:

Full conditional distributions for $(\mathbf{\Phi}, X_{\neg\text{obs}})$:

$$\mathbf{X}_{\neg\text{obs}}|\mathbf{\Phi}_E, \mathbf{X} : \quad \text{Poisson}\big((1 - p(\cdot))\Lambda(\cdot|\phi)\big)$$

$$\mathbf{\Phi}_E|\mathbf{X}_{\neg\text{obs}}, \mathbf{X} : \quad f(\phi|\mathbf{x}, \mathbf{x}_{\neg\text{obs}}) \propto f(\phi)\,\mathrm{e}^{-\int_W \Lambda(u|\phi)\mathrm{d}u} \prod_{u\in\mathbf{x}\cup\mathbf{x}_{\neg\text{obs}}} \Lambda(u|\phi)$$

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MCMC (Metropolis-within-Gibbs):

- $\mathbf{X}_{\neg\mathrm{obs}}|\mathbf{\Phi}_E, \mathbf{X}$: straightforward.
- $\mathbf{\Phi}_E|\mathbf{X}_{\neg\mathrm{obs}}, \mathbf{X}$: birth/death MCMC updates (Geyer & Møller 1994).

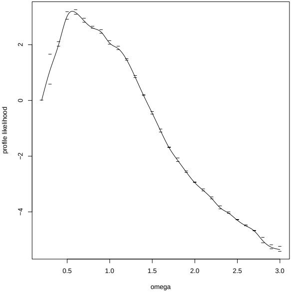
Maximization of likelihood

Likelihood based on all transects: multiply likelihoods for the different transects (approximately independent)

Maximize with respect to (κ, α) for finite set of ω values (Newton-Raphson)

Profile log likelihood function

$l_p(\omega) = \max_{\kappa,\alpha} \log L(\kappa, \alpha, \omega)$:

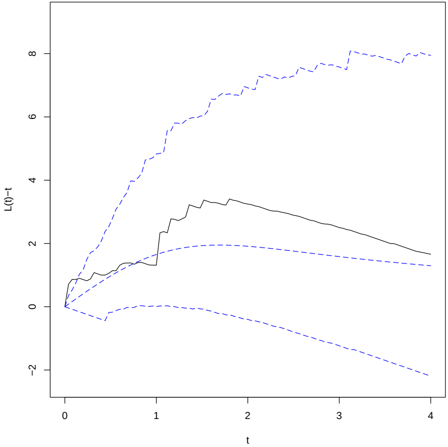


MLE: $\hat{\kappa} = 0.025$ $\hat{\alpha} = 2.4$ $\hat{\omega} = 0.6$.

95 % Confidence interval for whale intensity $\lambda = \kappa\alpha$: $[0.03; 0.08]$ (parametric bootstrap)

Model check using K-function

Plot based on $L(t) - t = \sqrt{K(t)/\pi} - t$



Summary

Estimating functions:

- computationally fast
- `R` packages available: `spatstat` and `InhomCluster`

Maximum likelihood estimation

- statistically more efficient
- long computations
- no standard software

References

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Møller, J. and Waagepetersen, R. (2003) *Statistical inference and simulation for spatial point processes*, Chapman & Hall/CRC Press.