Estimating functions for inhomogeneous spatial point processes with incomplete covariate data

Rasmus Waagepetersen
Department of Mathematics
Aalborg University
Denmark

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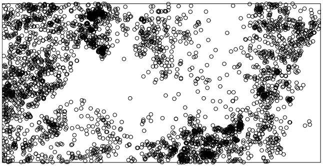
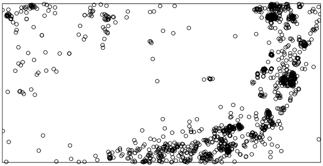
Data (Barro Colorado Island Forest Dynamics Plot)

Observation window: $S = [0, 1000] \times [0, 500]\text{m}^2$

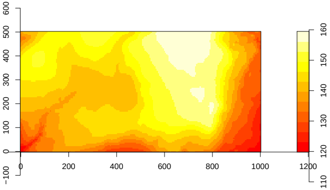
Beilschmiedia



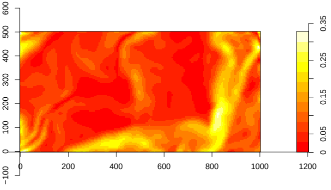
Ocotea



Elevation



Gradient norm (steepness)



Question: tree intensities related to elevation and gradient ?

Log-linear models for intensity function

$z(u) = \big(z_1(u), \ldots, z_p(u)\big)$ vector of covariates for each location u in observation window W.

E.g. $z(u) = (1, z_{\mathrm{elev}}(u), z_{\mathrm{grad}}(u))$ for rain forest example.

Log-linear model for intensity function:

$$\lambda(u; \beta) = \exp(z(u)\beta^{\mathsf{T}})$$

Interpretation:

$$\lambda(u; \beta)\mathrm{d}N_u = P(\text{point occurs in neighbourhood } N_u \text{ around } u)$$

Estimation of β

$\mathbf{X}$: point pattern observed in W

Poisson process log likelihood function:

$$l(\beta) = \sum_{u \in \mathbf{X}} z(u)\beta^{\mathsf{T}} - \int_W \lambda(u;\beta)\mathrm{d}u$$

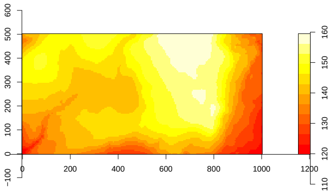
Poisson score estimating function:

$$u(\beta) = \sum_{u \in \mathbf{X}} z(u) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u$$

Also applicable for *non-Poisson* point processes with intensity function $\lambda(\cdot;\beta)$ (Schoenberg, 2005, Waagepetersen, 2007)

Missing covariate data

Elevation covariate



interpolated from elevation observations on grid.

However, evaluating

$$u(\beta) = \sum_{u \in \mathbf{X}} z(u) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u$$

requires $z(u)$ *observed* for any $u \in W$!

Approximations of log likelihood I

Suppose $z(u)$ observed at finite set of locations $\mathbf{Q} \subset W$.

Rathbun (1996) approximate

$$\int_W z(u)\lambda(u;\beta)\mathrm{d}u \approx \int_W \widehat{z(u)\lambda(u;\beta)}\mathrm{d}u$$

where $\widehat{z(u)\lambda(u;\beta)}$ unbiased prediction of $z(u)\lambda(u;\beta)$, $u \in W$ given $z(u)$, $u \in \mathbf{Q}$.

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Riemann approximation:

$$\int_W z(u)\lambda(u;\beta)\mathrm{d}u \approx \sum_{u \in \mathbf{Q}} w(u)z(u)\lambda(u;\beta)$$

where $w(u)$ quadrature weight for $u \in \mathbf{Q}$.

Approximations of log likelihood II: spatstat

Approximation of score function used in R package spatstat (Baddeley and Turner)

$$u(\beta) \approx u^{\text{spat}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{Q}} w(u) z(u) \lambda(u; \beta)$$

but now $\mathbf{Q} = \mathbf{X} \cup \mathbf{D}$ *includes observed points* in addition to 'dummy' points $\mathbf{D}$.

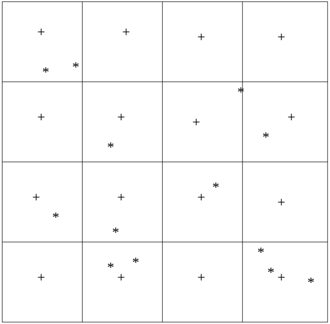
Note: $\mathbf{X}$ events of point pattern (supplied by 'nature')

$\mathbf{D}$ dummy points controlled by scientist.

Two types of weights: `grid` or `dirichlet`

Quadrature schemes in spatstat

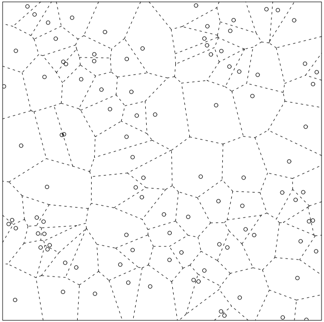
Grid



$$w(u) = \frac{|C_v|}{\#(\mathbf{X} \cap C_v) + 1}, \; u \in C_v$$

where $W = \cup_{v \in \mathbf{D}} C_v$

Dirichlet



$w(u)$ area of *Dirichlet cell* for u in Dirichlet tesselation generated by $\mathbf{Q}$.

spatstat: relation to generalized linear models and iterative weighted least squares

Estimating function

$$u^{\text{spat}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} w(u) z(u) \lambda(u; \beta)$$

formally equivalent to score function of weighted Poisson regression (log link):

$$u^{\text{spat}}(\beta) = \sum_{u \in \mathbf{X} \cup \mathbf{D}} w(u) z(u) \left[y_u - \lambda(u; \beta) \right]$$

with weights $w(u)$ and 'observations' $y_u = 1[u \in \mathbf{X}]/w(u)$.

Hence implementation straightforward using e.g. `glm()` in `R` (Poisson family, log link).

Distribution of parameter estimates from approximate score functions

?

Problem: hard to obtain distribution of parameter estimates from approximate score functions

Monte Carlo approximation of integral

Consider M *random* uniform dummy points $\mathbf{D}$ on W (*simple* random dummy points). Assume wlog $|W| = 1$.

Rathbun et al. (2006): Monte Carlo approx. of integral:

$$
\begin{aligned}
u^{rath}(\beta) &= \sum_{u \in \mathbf{X}} z(u) - \frac{1}{M} \sum_{u \in \mathbf{D}} z(u)\lambda(u;\beta) \\
&= u(\beta) + \Big[\frac{1}{M} \sum_{u \in \mathbf{D}} z(u)\lambda(u;\beta) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u\Big]
\end{aligned}
$$

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$$

CLT for Monte Carlo approximation:

$$
M^{1/2}\Big[\frac{1}{M} \sum_{u \in \mathbf{D}} z(u)\lambda(u;\beta) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u\Big] \xrightarrow{d} N(0, G)
$$

where G asymptotic covariance matrix.

Distribution of parameter estimate (Poisson process case)

Poisson score asymptotically normal

$$u(\beta) \approx N\big(0, i(\beta)\big)$$

where $i(\beta)$ Fisher information.

Suppose $\hat{\beta}$ solution of $u^{rath}(\beta) = 0$ where

$$u^{rath}(\beta) = u(\beta) + \Big[\frac{1}{M}\sum_{u \in \mathbf{D}} z(u)\lambda(u;\beta) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u\Big]$$

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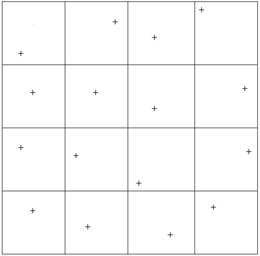
Then

$$\hat{\beta} \approx N\big(\beta, i(\beta)^{-1} + i(\beta)^{-1}\ G\ i(\beta)^{-1}\big)$$

NB: increasing intensity asymptotics: $N = \mathbb{E}\#\mathbf{X} \rightarrow \infty$! (fixed observation window W)

Stratified dummy points

Alternative: one uniformly sampled dummy point in each of M cells (stratified dummy points)



Suppose f continuously differentiable. Then CLT

$$M\left[\frac{1}{M}\sum_{u\in\mathbf{D}} f(u) - \int_W f(u)\mathrm{d}u\right] \xrightarrow{d} N(0, G)$$

(faster rate of convergence).

Asymptotic covariance matrix G depends on partial derivatives $\frac{\partial f_i}{\partial u_j}$ of f.

Monte Carlo versions of `spatstat` (Waagepetersen, 2007)

D point process of dummy points of intensity M. Monte Carlo version of `dirichlet`

$$u^{\text{dir}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} z(u) \frac{\lambda(u;\beta)}{\lambda(u;\beta) + M}$$

(either simple or stratified dummy points)

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(either simple or stratified dummy points)

Monte Carlo version of `grid`: (stratified dummy points)

$$u^{\text{grid}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \frac{1}{M} \sum_{u \in \mathbf{X} \cup \mathbf{D}} z(u)\lambda(u;\beta) \frac{1}{\#(\mathbf{X} \cap C_u) + 1}$$

Advantage: implementation using `glm()` just as for usual `spatstat`.

Distribution of parameter estimates (Poisson process case)

Grid: $u^{\text{grid}}(\beta)$ and $u^{\text{rath}}(\beta)$ with stratified dummy points same asymptotic distribution - hence same asymptotic distribution for parameter estimates.

Dirichlet: more subtle behaviour

- stratified dummy points: $u^{\text{dir}}(\beta)$ less efficient than $u^{\text{grid}}(\beta)/u^{\text{rath}}(\beta)$
- simple dummy points: asymptotic covariance matrix tends to that of MLE when $q = M/N$ increases ($N = \mathbb{E}\#\mathbf{X}$).

Numerical example: Poisson process

Case of Poisson process with covariate vector $(1, z_{\text{elev}})$ $\beta_{\text{elev}} = 0.1$.

Ratios of asymptotic standard errors for estimating function estimate $\hat{\beta}_{\text{elev}}$ relative to asymptotic standard error for MLE.

	simple					stratified			
q	0.25	1	10	100		0.25	1	10	100
u^{rath}	2.47	1.51	1.06	1.01	u^{grid}	1.08	1.01	1.00	1.00
u^{dir}	2.12	1.43	1.06	1.01	u^{dir}	1.56	1.53	1.53	1.53

$q = M/N$ proportion of dummy points.

u^{dir} better than Rathbuns Monte Carlo approximation u^{rath} but not useful in case of stratified dummy points.

Clustered point processes

Consider cluster process $\mathbf{X} = \cup_{c \in \mathbf{Y}} \mathbf{X}_c$ where $\mathbf{Y}$ stationary Poisson point process of intensity $\kappa > 0$.

Given $\mathbf{Y}$, clusters $\mathbf{X}_c$ are independent Poisson processes with intensity functions

$$\lambda_c(u) = \alpha \exp(z_{2:p}(u)\beta_{2:p}^{\mathsf{T}}) h(u - c; \omega)$$

Intensity function of $\mathbf{X}$ is then of log-linear form $\exp(z(u)\beta^{\mathsf{T}})$ where $\beta_1 = \log(\kappa\alpha)$ and $z_1(u) = 1$.

Estimate $\hat{\beta}$ using $u(\beta)$, $u^{\mathrm{dir}}(\beta)$ or $u^{\mathrm{grid}}(\beta)$ still asymptotically normal but covariance matrix now depends on $\psi = (\kappa, \omega)$.

Minimum contrast estimation of ψ

Closed form expression $K(t;\psi)$ and estimate

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X}}^{\neq} \frac{1[\|v-u\| \leq t]}{\lambda(u;\hat{\beta})\lambda(v;\hat{\beta})|W \cap W_{v-u}|}$$

available for K-function (extension to inhomogeneous case).

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Second step for estimation of ψ:

$$\hat{\psi} = \arg\min_{\psi} \int_0^r [\hat{K}(t)^c - K(t;\psi)^c]^2 \mathrm{d}t$$

NB: only requires $z(u)$ for $u \in \mathbf{X}$.

Inference regarding clustering

So far focused on intensity parameter β.

Sometimes clustering of main interest but still need to estimate β to adjust for aggregation due to covariates.

Joint asymptotic normality of $(\hat{\beta}, \hat{\psi})$ established in W & Guan (2007) for wide class of mixing point processes including

- cluster processes
- log Gaussian Cox processes (ψ covariance parameter of Gaussian field).

References

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