Asymptotic results for topological data analysis of point processes

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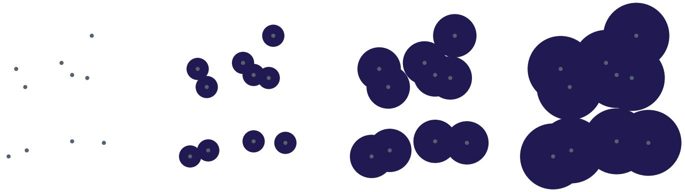
Topological data analysis

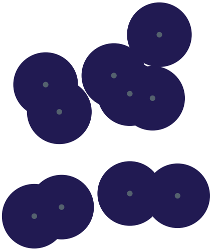
In 2D

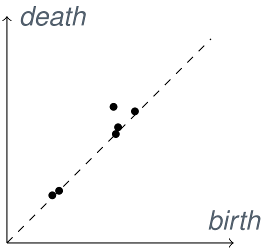


- $q = 0$ corresponds to connected components in the union of balls.
  - Interpretation: clusters.
  - A component dies when two components merge.
- $q = 1$ corresponds to connected components in the complement of the balls.
  - Interpretation: voids.
  - When a new component appears, we say that it is born.
  - When it gets covered again, we say it dies.

The persistence diagram

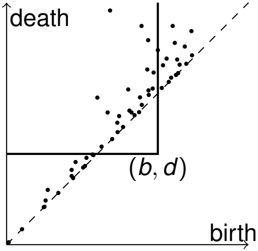
The information is summarized in the persistence diagram (q=1):





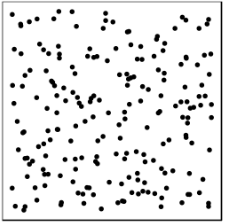
Persistent Betti numbers

- The persistent Betti numbers are given by the number of pairs born before *b* and dead after *d*.



Topological data analysis for point patterns

- TDA was invented for **deterministic** point patterns.
- Possibly measured with small amount of **noise**.
- In practice, many point patterns are realizations from a probability distribution.

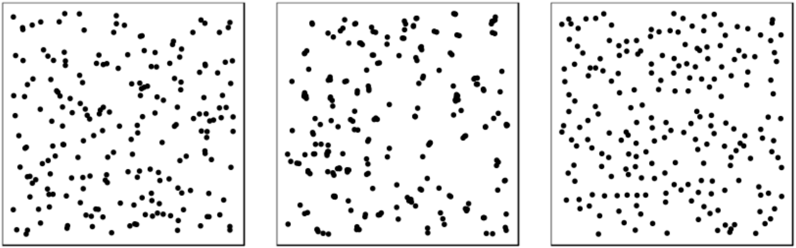


Point processes

A point process $\mathcal{P}$ is a random set of points in $\mathbb{R}^d$ which is a.s. locally finite:

- For any bounded $B \subseteq \mathbb{R}^d$, $\mathcal{P} \cap B$ is finite.

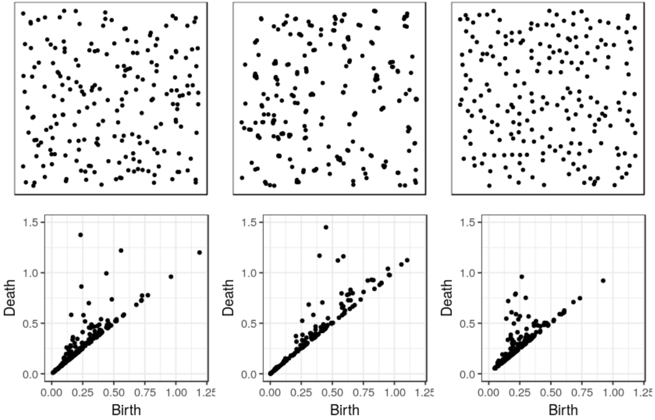
Examples:



Independence Clustering Repulsion

The persistence diagram

It seems that the persistence diagram can distinguish between different types of point patterns (q=1):



The Poisson process

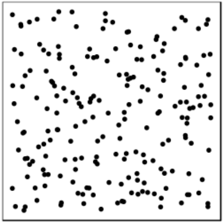
Complete spatial randomness

The (homogeneous) Poisson point process $\mathcal{P}$ is a point process on $\mathbb{R}^d$ given by:

1. For any bounded set $B \subseteq \mathbb{R}^d$:
$$\#(\mathcal{P} \cap B) \sim po(\rho|B|).$$
2. Given $\#(\mathcal{P} \cap B) = n$, the points in $\mathcal{P} \cap B$ are independent and uniformly distributed in B.

Property:

- If B_1 and B_2 are disjoint, then $\mathcal{P} \cap B_1$ and $\mathcal{P} \cap B_2$ are independent.



Poisson cluster process

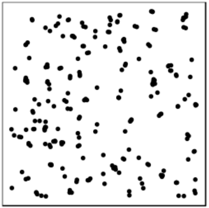
A clustered process

A Poisson cluster process is defined as follows:

- Start from a Poisson process $\mathcal{P}$.
- Replace each point in $\mathcal{P}$ by a random cluster (finite point set).

Matérn cluster process:

- Each cluster is Poisson distributed in $B_r(x)$.

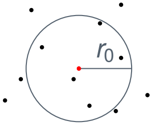


Gibbs point process

Often repulsive

For a finite point configuration $\eta \subseteq \mathbb{R}^d$, associate the energy $E(\eta)$.

- E is positive, increasing, translation + rotation invariant.
- Finite interaction range r_0 ("sufficiently small").



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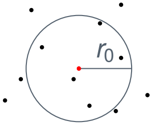


The finite volume Gibbs process on a bounded set $B \subseteq \mathbb{R}^d$ is absolutely continuous wrt. the Poisson process on B with intensity τ.

The density is

$$\eta \mapsto \exp(-\beta E(\eta))/Z(B).$$

Does not extend to all of $\mathbb{R}^d$!

Gibbs processes

Examples

Hardcore process:

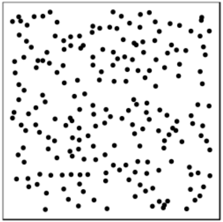
- If there exist $x, y \in \eta$ such that $|x - y| \leq r_0$ then $E(\eta) = \infty$.
- Otherwise $E(\eta) = 1$.

Strauss process:

$$E(\eta) = c_1 n_{r_0}(\eta),$$

where

- $n_{r_0}(\eta)$ is the number of pairs $x, y \in \eta$ with $0 < |x - y| \leq r_0$,
- $c_1 > 0$.



Statistics based on tda

Suppose we observe a point pattern P.

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3. Consider a derived functional, e.g.
   - $\sum_{(b,d)} f(b, d)$, for f "nice".
   - $APF_q(r) = \sum_{(b,d)}(d - b)\mathbb{1}\{b \leq r\}$, (Biscio/Møller, 2019).

Some limit theorems

- Suppose we observe $\mathcal{P}_n = \mathcal{P} \cap W_n$ in the window

$$W_n = \Big[-\frac{n^{1/d}}{2}, \frac{n^{1/d}}{2} \Big]^d.$$

- Only one realization available.
- Consider the limit when $n \to \infty$.

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Theorem (Hiraoka, Shirai, Trinh (2018))

For all point processes in this talk, there exists a constant $\beta_q^{b,d}$ such that

- $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\beta_q^{b,d}(\mathcal{P}_n) \to \beta_q^{b,d}$
- $\lim_{n\to\infty} \frac{1}{n}\beta_q^{b,d}(\mathcal{P}_n) \to \beta_q^{b,d}$ *a.s.*

Central limit theorems for persistent Betti numbers

Poisson case:
Yogeshwaran, Subag, Adler (2017), Hiraoka, Shirai, Trinh (2018):

$$\frac{\beta_q^{b,d}(\mathcal{P}_n) - \mathbb{E}[\beta_q^{b,d}(\mathcal{P}_n)]}{\sqrt{n}} \implies N(0, \sigma^2).$$

Gibbs case:
Hirsch, Otto, S. (in progress):

$$\frac{\beta_q^{b,d}(\mathcal{P}_n) - \mathbb{E}[\beta_q^{b,d}(\mathcal{P}_n)]}{\sqrt{n}} \implies N(0, \sigma^2).$$

Note: $\sigma^2 > 0$ whenever the energy E is finite.

Central limit theorems for persistence diagrams

M-bounded persistence pairs: only allow components with spatial diameter less than M.

Let $f : [0, T]^2 \to \mathbb{R}$ be bounded and define

$$\xi_f(P) = \sum_{(b,d)} f(b, d).$$

Theorem (Biscio, Chenavier, Hirsch, S. (2020))

Let $\mathcal{P}$ be a Poisson, Matérn cluster, or log-Gaussian point process in $\mathbb{R}^2$. Fix $M, T > 0$. Then,

$$\frac{\xi_f(\mathcal{P}_n) - \mathbb{E}[\xi_f(\mathcal{P}_n)]}{\mathrm{Var}(\xi_f(\mathcal{P}_n))} \to N(0, 1).$$

Functional CLTs

M-bounded persistent Betti-numbers: $\beta_{1,M}^{b,d}(\mathcal{P}_n)$ only counts components with spatial diameter less than *M*.

Theorem (Biscio, Chenavier, Hirsch, S. (2020), Biscio, S. (2022))

Let $\mathcal{P}$ be a Poisson, Matérn cluster, log-Gaussian or Gibbs point process in $\mathbb{R}^2$. Fix $M, T > 0$. Then,

$$\left\{ \frac{\beta_{1,M}^{b,d}(\mathcal{P}_n) - \mathbb{E}[\beta_{1,M}^{b,d}(\mathcal{P}_n)]}{\sqrt{n}} \right\}_{b,d \in [0,T]^2}$$

converges in Skorokhod topology to a centered Gaussian process on $[0, T]^2$.

Botnan, Hirsch (2023): Generalizations in the Poisson case.

Derived results

New CLTs can be derived from the functional CLT via the continuous mapping theorem.

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Example: The accumulated persistence function

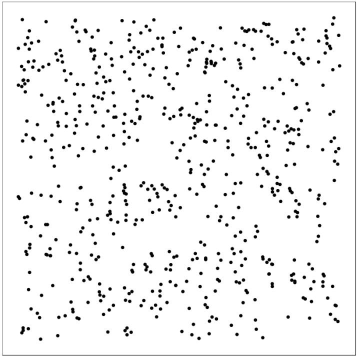
$$APF_q(r) = \sum_{(b,d)} (d - b)\mathbb{1}\{b \leq r\}.$$

Corollary

The process $\{\sqrt{n}(APF_q(r) - \mathbb{E}APF_q(r))\}_{r\in[0,R]}$ *converges to a centered Gaussian process.*

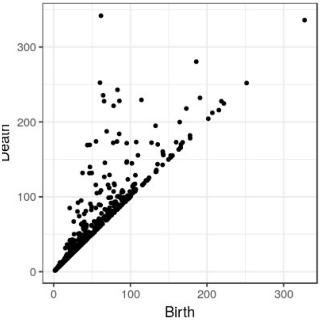
The minicolumn dataset (Nyengaard et al.)

- 634 brain neurons from human cerebral cortex
- Neurons tend to arrange in vertical columns
- 3D image projected to 2D
- Minicolumns would correspond to clusters in 2D

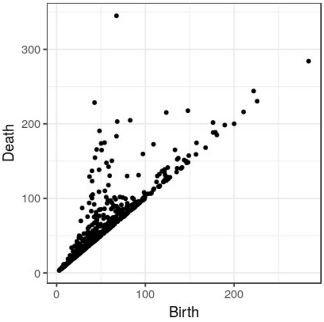


Persistence diagram for minicolumn data

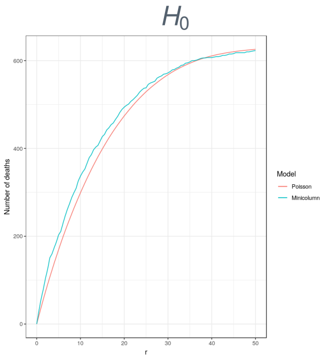
Dataset



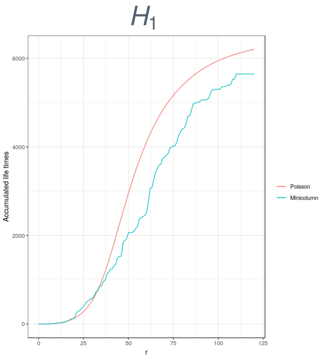
Poisson



Accumulated persistence function



$p = 0.0018$



$p = 0.00031$

About the proof

Rely on general CLTs for geometric functionals of the form:

- $H_n(\mathcal{P}_n) = \sum_{x \in \mathcal{P}_n} g(x, \mathcal{P}_n)$.

Gibbs processes:

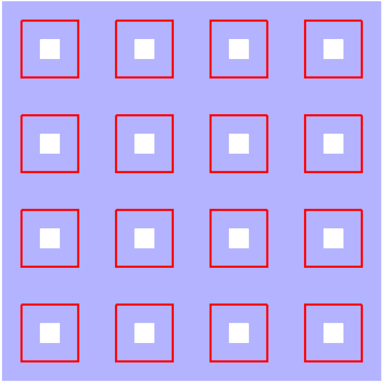
- CLT by Xia, Yukich (2015).

Other processes:

- CLT by Błaszczyszyn, Yogeshwaran, Yukich (2019)
- Requires variance is $\Omega(n)$

Variance lower bound

Idea of Xia and Yukich (2015), Biscio, Chenavier, Hirsch, S. (2020):



- Conditioned on the blue set and Λ, the red squares are independent.
- Lower bound on variance by sum of conditional variances of red squares.
- Each has conditional variance bounded from below.
- Number of red squares is $\Omega(n)$.

Summary

- TDA can be used to capture fine structure of a point pattern.
- CLTs are available, but still somewhat incomplete.
- Attempts to generalize to other random structures, e.g.
  - Random networks (Hirsch, Krebs 2022)
  - Random tesselations (Hirsch, Krebs, Redenbach 2023+)
- Disadvantages of TDA: results may be hard to interpret geometrically.

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Thank you for the attention!

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