Martingales - recap

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November 3, 2024

Continuous time martingale

Stochastic process $\{M(t)\}_{t \geq 0}$ is a martingale with respect to history $\mathcal{F}_t$ if

- M_t is measurable with respect to $\mathcal{F}_t$
- $\mathbb{E}[M_t | \mathcal{F}_u] = M_u$ when $u \leq t$

First property holds if $\mathbb{E}[\mathrm{d}M(t) | \mathcal{F}_{t-}] = 0$.

$\mathrm{d}M(t)$ is infinitesimal increment over infinitesimal time interval $[t, t + \mathrm{d}t[$

Independent and identically distributed survival times

Given survival data (T_i, Δ_i), $i = 1, \ldots, n$ define one-step counting processes

$$N_i(t) = 1[T_i \leq t, \Delta_i = 1] = 1[X_i \leq t, X_i \leq C_i]$$

and accumulated process

$$N(t) = \sum_{i=1}^{n} N_i(t).$$

$\mathcal{F}_t$: history up to time t (censoring, deaths, covariate information up to time t).

Define $Y_i(t) = 1[T_i \geq t]$. I.e. Y_i is one if ith individual at risk at time t and zero otherwise. Y_i is left-continuous and hence predictable.

$Y(t) = \sum_{i=1}^{n} Y_i(t)$ is the number at risk at time t.

Compensator

Define

$$\Lambda_i(t) = \int_0^t Y_i(u) h_i(u) \mathrm{d}u$$

where h is the hazard rate of X_i.

Then $\Lambda_i(t)$ is a continuous and hence predictable stochastic process.

Compensated counting process $M_i = N_i - \Lambda_i$ is a martingale.

Crucial result for martingale M

$$\tilde{M}(t) = \int_0^t K(u) \mathrm{d}M(u)$$

is a martingale if K is predictable.

Will use the above to show that score process of Cox partial likelihood is a martingale.

Nelson-Aalen

$$\Lambda(u) = Y(u)h(u)$$

$$\begin{aligned}
\hat{H}(t) = \sum_{t^* \in D : t^* \leq t} \frac{1}{Y(t^*)} &= \int_0^t \frac{1}{Y(u)} \mathrm{d}N(u) \\
&= \int_0^t \frac{1}{Y(u)} (\mathrm{d}N(u) - \Lambda(\mathrm{d}u)) + \int_0^t \frac{1}{Y(u)} \Lambda(\mathrm{d}u) \\
&= \int_0^t \frac{Y(u) 1[Y(u) > 0]}{Y(u)} \mathrm{d}h(u) + \int_0^t \frac{1}{Y(u)} \mathrm{d}M(u) \\
&\approx H(t) + \int_0^t \frac{1}{Y(u)} \mathrm{d}M(u)
\end{aligned}$$