Calculations with infinitesimal duration times

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Suppose X is a continuous random variable.

When we say that likelihood $f(x;\theta)$ is probability of observing x this does not really make sense since $P(X = x)$ is zero.

However, we can interpret 'observing x' as the event that X belongs to an infinitesimal interval $[x, x + \mathrm{d}x[$ in which case

$$P(X \in [x, x + \mathrm{d}x[) \approx f(x;\theta)\mathrm{d}x.$$

Ignoring the constant $\mathrm{d}x$, $f(x;\theta)$ becomes the likelihood.

Calculations involving infinitesimal increments can provide short cuts to many results in survival analysis.

Hazard function

The hazard function can be defined as the limit

$$\lambda(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} P(X \in [x, x + \Delta[| X \geq x)$$

Using differential calculus this becomes $f(x)/S(x)$.

A quick heuristic derivation is

$$\lambda(x)\mathrm{d}x = P(X \in [x, x+\mathrm{d}x[| X \geq x) = \frac{P(X \in [x, x + \mathrm{d}x[)}{S(x)} = \frac{f(x)\mathrm{d}x}{S(x)}$$

Likelihood for type II censoring

Suppose we observe (T_i, Δ_i), $i = 1, \dots, n$ in case of type II censoring. Let A denote the set of the r indices with $\Delta_i = 1$. That is for $i \in A$ we observe $X_i = T_i$ and for the remaining indices $l \in A^c$ we know $T_l \geq \max_{i \in A} T_i$.

Suppose wlog we observe $A = a = \{1, \dots, r\}$, $T_i = t_i$, $i \in a$ and that t_r is the maximal observed time. Then the likelihood is

$$
\begin{aligned}
& P(A = a, T_i \in [t_i, t_i + \mathrm{d}t[, i \in a) \\
= & P(X_1 \in [t_1, t_1 + \mathrm{d}t[, \dots, X_r \in [t_r, t_r + \mathrm{d}t[, X_l > \max_{i=1,\dots,r} X_i, l > r) \\
= & P(X_1 \in [t_1, t_1 + \mathrm{d}t[, \dots, X_r \in [t_r, t_r + \mathrm{d}t[, X_l > t_r, l = r+1, \dots, n) \\
= & \prod_{i=1}^{r} f_i(t_i)\mathrm{d}t \prod_{l=r+1}^{n} S_l(t_r)
\end{aligned}
$$

This can be written in the general form $\prod_{i=1}^{n} (\lambda_i(t_i)\mathrm{d}t)^{\delta_i} S_i(t_i)$