Wald and F-test

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Wald with rank-deficient covariance matrix

Suppose $X \sim N_n(m, \Sigma)$ where Σ does not have full rank. Suppose that $Km = 0$. Then

$$KX \sim N(0, K\Sigma K^{\mathsf{T}})$$

and

$$(K\Sigma K^{\mathsf{T}})^{-/2} KX \sim N(0, I_{p,n})$$

where $(K\Sigma K^{\mathsf{T}})^{-/2}$ generalized square root, $I_{p,n}$ has ones in the first p diagonal entries and zero elsewhere and p is rank of $K\Sigma K^{\mathsf{T}}$. Thus

$$(KX)^{\mathsf{T}}(K\Sigma K^{\mathsf{T}})^{-} KX \sim \chi^2(p)$$

Wald/F for linear model

Suppose $Y \sim N(\mu, \sigma^2 I)$ where $\mu \in L$. Then MLE

$$\hat{\mu} = PY \sim N(\mu, \sigma^2 P)$$

Consider hypothesis $\mu \in L' \subset L$. This is equivalent to that $(I - P')\mu = 0$. Thus with σ^2 known we could formulate Wald-test statistic

$$\begin{aligned} T(\sigma^2) =& \sigma^{-2}((I - P')PY)^{\mathsf{T}}[(I - P')P(I - P')]^{-}((I - P')PY) \\ =& \sigma^{-2}((P - P')Y)^{\mathsf{T}}(P - P')(P - P')Y = \sigma^{-2}\|(P - P')Y\|^2 \end{aligned}$$

which according to previous slide is $\chi^2(d - d')$ (recall generalized inverse of a projection matrix is the projection matrix itself).

With σ^2 replaced by REML estimate, $W(\tilde{\sigma}^2)/(d - d')$ is the F-statistic for the hypothesis.

Wald/F for two-way ANOVA

Consider the two-way ANOVA with systematic treatment factor T and random factors P and $P \times T$.

Then covariance matrix can be decomposed as

$$\Sigma = \lambda_P P_P + \lambda_{P\times T}\tilde{Q}_{P\times T} + \sigma^2 \tilde{Q}_I.$$

If we want to test no treatment effect this is equivalent to that $Q_T Y = P_T Y - P_0 Y = 0$. Also note that $Q_T \Sigma Q_T = \lambda_{P\times T} Q_T$.

Hence, Wald test becomes

$$T = (Q_T Y)^\mathsf{T}(Q_T \Sigma Q_T)^{-1} Q_T Y = \|Q_T Y\|^2/\tilde{\lambda}_{P\times T} = F \cdot (d_T - 1).$$