19.2.3 b)

$$x_{t+1} - x_t - \frac{1}{3} y_t = -1 \qquad x_0 = 5$$
$$x_{t+1} + y_{t+1} - \frac{1}{6} y_t = 8\frac{1}{2} \qquad y_0 = 4$$

$$u_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} \qquad u_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u_{t+1} + \begin{bmatrix} -1 & -\frac{1}{3} \\ 0 & -\frac{1}{6} \end{bmatrix} u_t = \begin{bmatrix} -1 \\ 8\frac{1}{2} \end{bmatrix}$$

Partikulær løsning: $u_t = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -\frac{1}{3} \\ 0 & -\frac{1}{6} \end{bmatrix} \right) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8\frac{1}{2} \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8\frac{1}{2} \end{bmatrix} \qquad k_2 + 3 - \frac{1}{6}$$

$\downarrow$

det $-\frac{1}{3} \neq 0$ ✓

$$k_2 = 3 \qquad k_1 + 3\frac{5}{6} = \frac{17}{2} \Leftrightarrow k_1 = \frac{17}{2} - \frac{5}{2} \checkmark$$

$$\Leftrightarrow k_1 = 6 \checkmark$$

Homogen ligning: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u_{t+1} + \begin{bmatrix} -1 & -\frac{1}{3} \\ 0 & -\frac{1}{6} \end{bmatrix} u_t = 0$

Gæt $u_t = \begin{bmatrix} n b^t \\ m b^t \end{bmatrix}$

$$b^t \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} b + \begin{bmatrix} -1 & -\frac{1}{3} \\ 0 & -\frac{1}{6} \end{bmatrix} \right) \begin{bmatrix} n \\ m \end{bmatrix} = 0$$

Ikke-triviel løsning hvis $\begin{vmatrix} b-1 & -\frac{1}{3} \\ b & b-\frac{1}{6} \end{vmatrix} = 0$

$$(b-1)(b-\tfrac{1}{6}) + \tfrac{1}{3}b = b^2 - \tfrac{1}{6}b - b + \tfrac{1}{6} + \tfrac{1}{3}b$$

$$= b^2 - \tfrac{5}{6}b + \tfrac{1}{6} = 0$$

$$D = \left(\tfrac{5}{6}\right)^2 - \tfrac{4}{6} = \tfrac{25}{26} - \tfrac{24}{36} = \tfrac{1}{36} \qquad \sqrt{D} = \tfrac{1}{6}$$

$$b_1 = \frac{\frac{5}{6} - \frac{1}{6}}{2} = \frac{1}{3} \qquad b_2 = \frac{\frac{5}{6} + \frac{1}{6}}{2} = \frac{1}{2}$$

Find n, m:

$b_1 = \frac{1}{3}$

a) $\begin{bmatrix} \frac{1}{3}-1 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3}-\frac{1}{6} \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & +\frac{1}{6} \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} = 0$

$\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & +\frac{1}{6} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & +\frac{1}{6} \end{bmatrix} \qquad \frac{1}{3}n + \frac{1}{6}m = 0 \Leftrightarrow n = -\frac{1}{2}m$

b) $b_2 = \frac{1}{2}$

$$\begin{matrix} -\frac{1}{2} & -\frac{1}{3} \\ \frac{1}{2} & \frac{2}{6} \end{matrix} \sim \begin{matrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{3} \end{matrix} \qquad \frac{1}{2}n + \frac{1}{3}m = 0 \iff$$

$$n = -\frac{2}{3}m$$

Da

$$u_t = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}m_1 \\ m_1 \end{bmatrix} \frac{1}{3}^t + \begin{bmatrix} -\frac{2}{3}m_2 \\ m_2 \end{bmatrix} \frac{1}{2}^t$$

$x_0 = 5 \qquad y_0 = 4$:

$$\begin{aligned} 6 - \frac{1}{2}m_1 - \frac{2}{3}m_2 &= 5 \\ 3 + m_1 + m_2 &= 4 \end{aligned} \iff \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} -\frac{1}{2} & -\frac{2}{3} & -1 \\ 1 & 1 & 1 \end{array}\right] \sim \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -1 \\ 0 & -\frac{1}{3} & -1 \end{bmatrix}$$

$m_2 = 3 \qquad -\frac{1}{2}m_1 - 2 = -1 \iff -\frac{1}{2}m_1 = 1 \iff m_1 = -2$

$$u_t = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \frac{1}{3}^t + \begin{bmatrix} -2 \\ 3 \end{bmatrix} \frac{1}{2}^t$$

Check

$$\cancel{6} + \frac{1}{3}^{t+1} - 2\,\frac{1}{2}^{t+1} - \cancel{6} - \frac{1}{3}^{t} + 2\,\frac{1}{2}^{t} - \frac{1}{3}\left(3 - 2\,\frac{1}{3}^{t} + 3\,\frac{1}{2}^{t}\right)$$

$$= \frac{1}{3}^{t}\left(\frac{1}{3} - 1 + \frac{2}{3}\right) + \frac{1}{2}^{t}\left(-1 + 2 - 1\right) - \frac{1}{3}\cdot 3 = -1 \quad \checkmark$$

$$6 + \frac{1}{3}^{t+1} - 2\,\frac{1}{2}^{t+1} + 3 - 2\,\frac{1}{3}^{t+1} + 3\,\frac{1}{2}^{t+1} - \frac{1}{6}\left(3 - 2\,\frac{1}{3}^{t} + 3\,\frac{1}{2}^{t}\right)$$

$$= 8\frac{1}{2} + \frac{1}{3}^{t}\left(\frac{1}{3} - \frac{2}{3} + \frac{2}{6}\right) + \frac{1}{2}^{t}\left(-1 + \frac{3}{2} - \frac{3}{6}\right) = 8\frac{1}{2} \quad \checkmark$$