Separable variable

Ligning $\quad f(y)\dfrac{dy}{dt} = h(t) \qquad (*)$

Sæt $\quad y(t) = g(t)$

Da er ligningen $\quad f(g(t))g'(t) = h(t) \iff$

$$\int f(g(t))g'(t)\,dt = \int h(t)\,dt \iff F(g(t)) = \int h(t)\,dt .$$

Hvis F har invers fkt F^{-1} er $g(t) = F^{-1}\left(\int h(t)\,dt\right)$.

En bekvem omskrivning af $(*)$ er

$$f(y)\frac{dy}{dt} = h(t) \quad \text{"}\!\iff\!\text{"} \quad f(y)\,dy = h(t)\,dt \iff \int f(y)\,dy = \int h(t)\,dt$$

$$\Downarrow$$

(strengt taget ikke korrekt da $\frac{dy}{dt}$ er en funktion og ikke en brøk)

$$F(g(t)) = \int h(t)\,dt .$$

Ex

$$2t\,dy - y\,dt = 0 \quad 2t\,dy = y\,dt \overset{t,y\neq 0}{\iff} \frac{2}{y}\,dy = \frac{1}{t}\,dt$$

$$\iff 2\int \frac{1}{y}\,dy = \int \frac{1}{t}\,dt \iff 2\ln y = \ln t + c \iff$$

$$\ln y^2 = \ln t + c \iff y^2 = t\cdot \underbrace{\exp(c)}_{A} \iff y = \sqrt{At}$$

(antager her $y \geq 0$)

Check: $\quad \dfrac{dy}{dt} = \dfrac{A}{2\sqrt{t}} \qquad 2\dfrac{1}{y}\dfrac{dy}{dt} = \dfrac{2}{\sqrt{At}}\,\dfrac{A}{2\sqrt{At}} = \dfrac{1}{t}$ ✓.

Ex

$$3y^2\,dy - t\,dt = 0 \iff 3y^2\,dy = t\,dt$$

$$\iff 3\int y^2\,dy = \int t\,dt \iff y^3 = \frac{1}{2}t^2 + c$$

$$\iff y = \sqrt[3]{\frac{1}{2}t^2 + c} = \left(\frac{1}{2}t^2+c\right)^{\frac{1}{3}}$$

Check

$$\frac{dy}{dt} = \frac{1}{3}\left(\frac{1}{2}t^2+c\right)^{-\frac{2}{3}} t$$

$$3y^2\frac{dy}{dt} = 3\left(\frac{1}{2}t^2+c\right)^{\frac{2}{3}} \frac{1}{3}\left(\frac{1}{2}t^2+c\right)^{-\frac{2}{3}} t = t \quad \checkmark$$

Logistisk vekst (som opgave)

$$\frac{dy}{dt} = k\left(1-\frac{y}{A}\right)y \iff \frac{1}{k\left(1-\frac{y}{A}\right)y}\,dy = 1\,dt$$

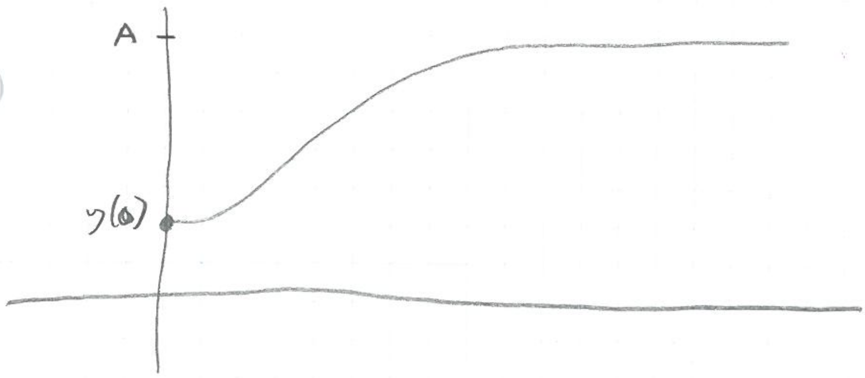
$$\iff \frac{1}{k}\int \frac{1}{y} + \frac{\frac{1}{A}}{\left(1-\frac{y}{A}\right)}\,dy = \int 1\,dt \iff$$

$$\ln y - \ln\left(1-\frac{y}{A}\right) = kt + C \iff \ln\frac{y}{1-\frac{y}{A}} = kt + c$$

$$\iff \frac{y}{1-\frac{y}{A}} = B\exp(kt) \iff y = B\exp(kt) - \frac{B}{A}\,y\exp(kt) \iff$$

$$y = \frac{B\exp(kt)}{1+\frac{B}{A}\exp(kt)} \iff y = \frac{1}{\frac{1}{B}\exp(-kt)+\frac{1}{A}}$$

(dvs $y(t) \to A$ når $t \to \infty$) $\quad y(0) = \dfrac{B}{1+\frac{B}{A}}$



Bernoulli-equation

$$\frac{dy}{dt} + Ry = Ty^m \qquad (m \neq 0, 1) \qquad (T \neq 0) \quad (R \neq 0)$$

$$\Leftrightarrow \quad y^{-m}\frac{dy}{dt} + Ry^{1-m} = T$$

$$\left[Z = y^{1-m} \qquad \frac{dz}{dt} = (1-m)y^{-m}\frac{dy}{dt} \right]$$

$$\Leftrightarrow \quad \frac{1}{1-m}\frac{dz}{dt} + Rz = T \quad \Leftrightarrow \quad \frac{dz}{dt} + \underbrace{(1-m)R}_{u}z = \underbrace{(1-m)T}_{w}$$

Dvs 1. ordens lineær differentialligning mht z.

Vha generel løsningsformel:

$$z(t) = \exp\left(-(1-m)\int R(t)\,dt\right)A + \exp\left(-(1-m)\int R(t)\,dt\right)\int (1-m)T(t)\exp\left((1-m)\int R(t)\,dt\right)dt$$

Finally $\quad y(t) = z(t)^{\frac{1}{1-m}}$.

Ex

$$\frac{dy}{dt} + \frac{1}{t} y = y^3 \qquad R(t) = \frac{1}{t} \quad T(t) = 1 \quad m = 3$$

$$(1-m)\int R(t)\,dt = -2\ln(t)$$

$$z(t) = \exp(-(-2\ln(t)))A + \exp(-(-2\ln(t)))\int -2\cdot 1 \exp(-2\ln(t))\,dt$$

$$= \exp(\ln t^2)A + \exp(\ln t^2)\int -2 \exp(\ln t^{-2})\,dt$$

$$= At^2 + t^2 \int -2t^{-2}\,dt = At^2 + t^2\, 2t^{-1}$$

$$= At^2 + 2t$$

$$y(t) = z(t)^{-\frac{1}{2}} = \frac{1}{\sqrt{At^2+2t}}$$

Check

$$\frac{dy}{dt} = -\frac{1}{2} z(t)^{-\frac{3}{2}} z'(t) = -\frac{1}{2} y^3 (2At+2)$$

$$\frac{dy}{dt} + \frac{1}{t} y = -\frac{1}{2} y^3 (2At+2) + \frac{1}{t}(At^2+2t)^{-\frac{1}{2}}$$

$$= y^3\left(-\frac{1}{2}(2At+2) + \frac{1}{t} y^{-2}\right)$$

$$= y^3\left(-\frac{1}{2}(2At+2) + \frac{1}{t}(At^2+2t)\right) = y^3(-At-1+At+2)$$

$$= y^3 \checkmark$$

Solows vækstmodel

$$\boxed{Q = \text{production} = f(K,L). \text{ Antager } f_K, f_L > 0 \text{ og } f_{KK}, f_{LL} < 0.}$$

Antager f lineært homogen, dvs $f(rK, rL) = r f(K,L)$

Med $r = \frac{1}{L}$ fås da $f\left(\frac{K}{L}, \frac{L}{L}\right) = \frac{1}{L} f(K,L) \iff$

$$f(K,L) = L f\left(\frac{K}{L}, 1\right) = L \varphi(k), \quad k = \frac{K}{L}$$

$$f_K = \frac{\partial}{\partial K} f(K,L) = L \frac{d}{dK} \varphi(k) = L \varphi'(k) \frac{1}{L} = \varphi'(k)$$

$$f_{KK} = \frac{\partial}{\partial K} \varphi'(k) = \varphi''(k) \frac{1}{L}$$

Dermed $\varphi'(k) > 0$ og $\varphi''(k) < 0$.

Yderligere antagelser: $\frac{dK}{dt} = sQ$

(angiver K og L udvikler sig over tid) $\frac{dL/dt}{L} = \lambda \quad (\Rightarrow L(t) = A e^{\lambda t})$

$$K = kL \Rightarrow \frac{dK}{dt} = \frac{dk}{dt} L + k \frac{dL}{dt}$$

Samtidigt $\frac{dK}{dt} = sQ = sL\varphi(k)$ og $\frac{dL}{dt} = \lambda L$

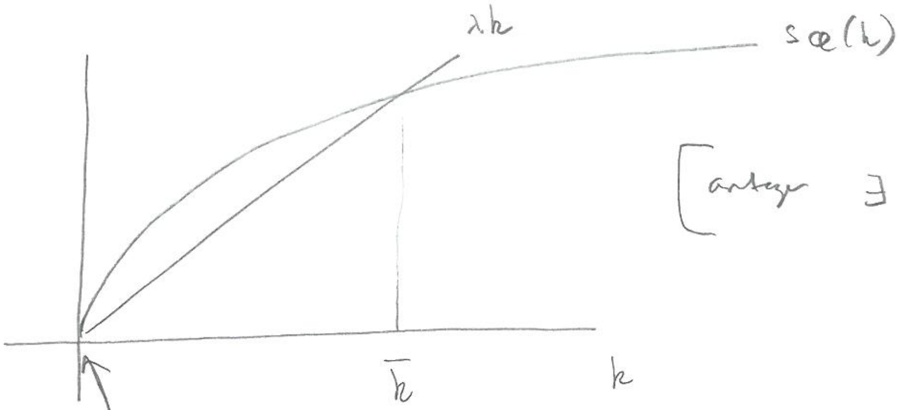
Dvs $\frac{dk}{dt} L + k \lambda L = sL\varphi(k) \overset{L>0}{\iff} \frac{dk}{dt} + \lambda k = s\varphi(k)$

Dvs. differentialligning for k.

Hvad er ligevægtstilstanden for $k = \frac{K}{L}$?

$$\frac{dk}{dt} = s\varphi(k) - \lambda k$$

Kvalitativ analyse

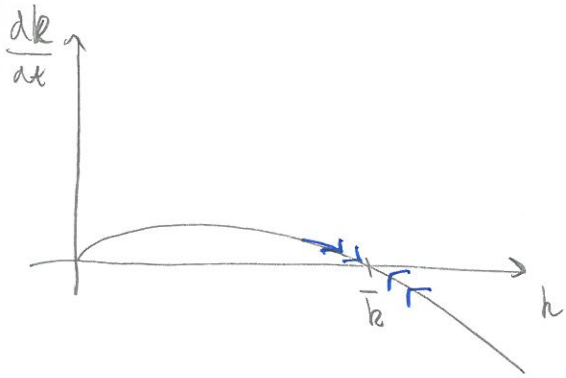


$(\varphi'(k) > 0 \quad \varphi''(k) < 0)$

[antager $\exists\, k$ så $s\varphi(k) > \lambda k$]

antager
Q nul hvis
K nul

her er $\frac{dk}{dt} = 0$



Hvis $k < \bar{k}$ vil

$\frac{dk}{dt} > 0$ dvs k vokser

og omvendt hvis $k > \bar{k}$

I ligevægt vil $k = \frac{K}{L} = \bar{k}$ og derefter vil

K altså vokse proportionalt med L

Og endvidere $Q = L \underbrace{\varphi(\bar{k})}_{\text{konstant}}$

Explicit model for Q (Cobb-Douglas)

$$Q = K^{\alpha} L^{1-\alpha} = L \underbrace{k^{\alpha}}_{q(k)} \qquad 0 \leq \alpha \leq 1$$

$$\frac{dk}{dt} = s k^{\alpha} - \lambda k \iff \frac{dk}{dt} + \lambda k = s k^{\alpha}$$ (Bernoulli equation)

Dividere med k^{α} og gang med $(1-\alpha)$

$$(1-\alpha) k^{-\alpha} \frac{dk}{dt} + \lambda k^{1-\alpha}(1-\alpha) = s(1-\alpha) \qquad (*)$$

$$z = k^{1-\alpha} \qquad \frac{dz}{dt} = (1-\alpha) k^{-\alpha} \frac{dk}{dt}$$

(første ordens lineær konstant koeff og høyreside)

$(*)$ svarer så til $$\frac{dz}{dt} + \underbrace{(1-\alpha)\lambda}\, z = \underbrace{s(1-\alpha)}$$

$$z(t) = A e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda} \qquad A = \left[z(0) - \frac{s}{\lambda}\right]$$

Dvs $$k^{1-\alpha} = \left[z(0) - \frac{s}{\lambda}\right] e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

Dvs $\alpha < 1 \Rightarrow k^{1-\alpha} \longrightarrow \frac{s}{\lambda}$ eller $k \to \left(\frac{s}{\lambda}\right)^{\frac{1}{1-\alpha}} = \bar{k}$

Dvs igjen konklusjon $k = \frac{K}{L} \to \bar{k}$ konstant forhold

Udledning i makro-kursus

$c(k) = k^{\alpha}$

(ligevægt)

$\frac{dk}{dt} + \lambda k = s k^{\alpha}$. dvs $\frac{dk}{dt} = 0 \iff \lambda k = s k^{\alpha}$

$\iff k^{1-\alpha} = \frac{s}{\lambda}$

$\iff \underline{\underline{k = \sqrt[1-\alpha]{\frac{s}{\lambda}}}}$

ligevægts-værdi.

Ved løsning af Bernoulli-ligning fandt vi

$k(t)$ for alle t:

$$k(t) = \sqrt[1-\alpha]{A e^{-\lambda(1-\alpha)t} + \frac{s}{\lambda}}$$

og kunne da observere $k(t) \to \sqrt[1-\alpha]{\frac{s}{\lambda}}$ når

$t \to \infty$.