2. ordens differensligninger

2. ordens differens $\Delta^2 y_t$ er diskret analog

til $\frac{d^2y}{dt^2}$:

$$\frac{d^2y}{dt^2} = \lim_{h_1 \to 0} \frac{y'(t+h_1) - y'(t)}{h_1} = \lim_{h_1 \to 0} \frac{1}{h_1}\left[y'(t+h_1) - y'(t)\right]$$

$$= \lim_{h_1 \to 0} \frac{1}{h_1}\left[\lim_{h_2 \to 0}\left[\frac{y(t+h_1+h_2) - y(t+h_1)}{h_2} - \frac{y(t+h_2) - y(t)}{h_2}\right]\right]$$

$$= \lim_{h_1 \to 0} \lim_{h_2 \to 0} \frac{1}{h_1 h_2}\left[y(t+h_1+h_2) - y(t+h_1) - y(t+h_2) + y(t)\right]$$

$$\overset{h_1 = h_2 = h}{=} \lim_{h \to 0} \frac{1}{h^2}\left[\underbrace{y(t+2h) - 2y(t+h) + y(t)}_{\Delta_h^2 y_t}\right]$$

Sætter vi $h = 1$ tidsenhed fås

$$\Delta_h^2 y_t = \Delta_1^2 y_t = \Delta^2 y_t = y(t+2) - 2y(t+1) + y(t)$$

Hvis vi inddrager 2. ordens differenser i en differensligning, svarer det til at ligning indeholder både y_t, y_{t+1} og y_{t+2}

2. ordens lineær differensligning

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = C$$

Som sædvanlig: løsning $y_t = y_{pt} + y_{ct}$

Finder y_p:

a) Gæt $y_{pt} = k$. Indsætter: $k + a_1 k + a_2 k = C \iff$

$$k(1 + a_1 + a_2) = c.$$

Dvs hvis $(1 + a_1 + a_2) \neq 0$ fås $y_{pt} = k = \dfrac{c}{1 + a_1 + a_2}$

Antag $1 + a_1 + a_2 = 0 \iff a_1 + a_2 = -1 \iff a_2 = -1 - a_1$

b) Gæt $y_{pt} = kt$. Indsætter: $k(t+2) + a_1 k(t+1) - (1+a_1)kt = c$

$$\iff 2k + \underline{kt} + \underline{a_1 kt} + a_1 k - \underline{kt} - \underline{a_1 kt} = c$$

$$\iff 2k + a_1 k = C \iff k(2 + a_1) = c.$$

Dvs. hvis $a_1 \neq -2$ fås $y_{pt} = kt = \dfrac{c}{2 + a_1} t$.

c) $a_1 + a_2 = -1$ og $a_1 = -2$. Dvs $a_2 = 1$

Dermed er ligningen $\underbrace{y_{t+2} - 2y_{t+1} + y_t}_{\Delta^2 y_t.} = C$

Indsætter: $k(t+2)^2 - 2k(t+1)^2 + kt^2 = c \iff$

$$kt^2 + k4t + 4k - 2kt^2 - 4kt - 2k + kt^2 = c \iff$$

$$2k = c \iff k = \frac{c}{2}.$$

Dvs i tilfælde c: $y_{pt} = \frac{c}{2} t^2$.

Løsning af homogene ligning (y_c)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

Gæt: $y_t = Ab^t$ (eksponentialfunktion)

Indsætter:

$$Ab^{t+2} + a_1 Ab^{t+1} + a_2 Ab^t = 0 \iff$$

$$Ab^t(b^2 + a_1 b + a_2) = 0 \iff b^2 + a_1 b + a_2 = 0.$$

Dvs helt ækvivalent med 2. ordens differential-ligninger fås 3 tilfælde:

2 reelle rødder, reel dobbeltrod, 2 komplekse rødder,

I) 2 reelle rødder. $D = a_1^2 - 4a_2 > 0$.

$$b_1 = \frac{-a_1 + \sqrt{D}}{2} \qquad b_2 = \frac{-a_1 - \sqrt{D}}{2}$$

Dvs 2 løsninger $A_1 b_1^t$ og $A_2 b_2^t$ som kombineres til en generel løsning

$$y_{ct} = A_1 b_1^t + A_2 b_2^t.$$

(samme argumenter som for differentialligninger)

(En beg. betingelse ikke nok til at fastlægge - omvendt, to beg. betingelser kræver to koefficienter A_1 og A_2).

<u>Eksempel</u>

$y_{t+2} + y_{t+1} - 2y_t = 12 \qquad y_0 = 4$ og $y_1 = 5$

$a_1 + a_2 = 1 - 2 = -1$. Dvs $y_{pt} = \frac{12}{1+2} t = 4t$.

$D = 1 - 4(-2) = 9 > 0$

$b_1 = \frac{-1+3}{2} = 1. \qquad b_2 = \frac{-1-3}{2} = -2.$

$y_t = A_1 + A_2(-2)^t + 4t$

$$\left.\begin{array}{l} y_0 = A_1 + A_2 = 4 \\ y_1 = A_1 + -2A_2 + 4 = 5 \end{array}\right\} \Leftrightarrow \begin{array}{l} A_1 + A_2 = 4 \\ 3A_1 + 4 = 13 \end{array} \Leftrightarrow \begin{array}{l} A_2 = 1 \\ A_1 = 3 \end{array}$$

$y_t = 3 + (-2)^t + 4t$. (oscillerende, divergent)

II) $\underline{D=0}$

$$b = \frac{-a_1}{2}$$

Analogt med 2. orden differentialligninger:

$$y_t = A_1 b^t + A_2 t b^t$$

$\underline{\text{Ex}}$

$$y_{t+2} + 4y_{t+1} + 4y_t = 9 \qquad y_0 = 5 \qquad y_1 = 3$$

$a_1 + a_2 \neq -1$

$$y_{pt} = \frac{9}{9} = 1.$$

$$D = 4^2 - 4 \cdot 4 = 0 \qquad b = \frac{-4}{2} = -2$$

$$y_t = 1 + A_1(-2)^t + A_2 t(-2)^t$$

$$\begin{aligned} y_0 &= 1 + A_1 = 5 \\ y_1 &= 1 - 2A_1 - 2A_2 = 3 \end{aligned} \iff \begin{aligned} A_1 &= 4 \\ -2A_2 &= 3 - 1 + 8 \end{aligned} \iff \begin{aligned} A_1 &= 4 \\ A_2 &= -5 \end{aligned}$$

Dvs $y_t = 1 + 4(-2)^t - 5t(-2)^t$

(oscillerende, divergent)

III) $D<0$ (2 komplekse rødder)

$$r_1 = \frac{-a_1 + i\sqrt{|D|}}{2} = h + iv$$

$$r_2 = h - iv$$

$$y_{ct} = A_1 (h+iv)^t + A_2 (h-iv)^t$$

Omskriver til polære koordinater:

$$(h, v) = R(\cos\theta, \sin\theta) \qquad \theta = \tan^{-1}\left(\frac{v}{h}\right) \quad R = \sqrt{h^2+v^2}$$

$$(h+iv) = R(\cos\theta + i\sin\theta) = R\exp(i\theta)$$

Dvs $(h+iv)^t = (R\exp(i\theta))^t = R^t \exp(i\theta t)$

$$= R^t(\cos\theta t + i\sin\theta t)$$

Tilsvarende $(h-iv)^t = R^t[\cos\theta t - i\sin\theta t]$

Dvs $y_{ct} = (A_1 + A_2)R^t\cos\theta t + i(A_1 - A_2)R^t \sin\theta t$

$$= A_5 R^t \cos\theta t + i A_6 R^t \sin\theta t.$$

Igen kan vi omskrive vha polære koordinater

$(A_5, A_6) = (A\cos\varepsilon, A\sin\varepsilon)$ og additionsformler:

$$y_{ct} = AR^t\cos(\theta t - \varepsilon).$$

Ex

$$y_{t+2} + \frac{1}{4} y_t = 5 \qquad (a_1 = 0 \quad a_2 = \frac{1}{4})$$

$$y_{pt} = \frac{5}{1 + 0 + \frac{1}{4}} = \frac{5}{\frac{5}{4}} = 4.$$

$$D = 0 - 4 \cdot \frac{1}{4} = -1 \qquad r_1 = \frac{0 + i1}{2} = i\frac{1}{2} \qquad r_2 = -i\frac{1}{2}$$

$$R = \sqrt{\frac{1}{2}^2} = \frac{1}{2} \qquad \tan\theta = \infty \iff \theta = \frac{\pi}{2}$$

$(\tan\theta = \frac{\sin\theta}{\cos\theta}$ dvs ∞ hvis $\cos\theta = 0$, dvs $\theta = \frac{\pi}{2})$

Dvs $y_t = 4 + \frac{1}{2}^t \left(A_5 \cos\frac{\pi}{2}t + A_6 \sin\frac{\pi}{2}t\right)$.

Beg. betingelser $y_0 = 5$ $\quad y_1 = 3$

$$\left.\begin{aligned} y_0 &= 4 + A_5 + A_6 \cdot 0 = 5 \\ y_1 &= 4 + \tfrac{1}{2} A_5 \cdot 0 + \tfrac{1}{2} A_6 \cdot 1 = 3 \end{aligned}\right\} \iff$$

$$\begin{aligned} A_5 &= 1 \\ 4 + \tfrac{1}{2} A_6 &= 3 \end{aligned} \iff \begin{aligned} A_5 &= 1 \\ A_6 &= -2. \end{aligned}$$

Dempet oscillerende $y_t \rightarrow 4$ når $t \rightarrow \infty$.

Samuelson model

$$Y_t = C_t + I_t + G_0$$ (BNP = forbrug + invest. + offentlige udgifter)

$$C_t = \gamma Y_{t-1} \qquad 0 < \gamma < 1$$

$$I_t = \alpha (C_t - C_{t-1}) \qquad \alpha > 0$$

$$\left(I_t = \alpha\gamma (Y_{t-1} - Y_{t-2}) \right)$$

$$Y_t = C_t + I_t + G_0 \iff$$

$$Y_t = \gamma Y_{t-1} + \alpha\gamma (Y_{t-1} - Y_{t-2}) + G_0 \iff$$

$$Y_t - \gamma(1+\alpha) Y_{t-1} + \alpha\gamma Y_{t-2} = G_0$$

Skifter tidsindex 2 frem:

$$\boxed{Y_{t+2} - \gamma(1+\alpha) Y_{t+1} + \alpha\gamma Y_t = G_0}$$

$$1 + a_1 + a_2 = 1 - \gamma(1+\alpha) + \alpha\gamma = 1 - \gamma > 0 \ !$$

Dvs $\quad y_p = \dfrac{G_0}{1-\gamma}$.

$D = \gamma^2 (1+\alpha)^2 - 4\alpha\gamma$. Dvs $D > 0 \iff$

$$\gamma^2 (1+\alpha)^2 > 4\alpha\gamma \iff \gamma > \frac{4\alpha}{(1+\alpha)^2} .$$

Ex $\alpha = 1$: $\frac{4\alpha}{(1+\alpha)^2} = \frac{4}{2^2} = 1 > \gamma$!

Da løsning på formen

$$Y_t = \frac{G_0}{1-\gamma} + AR^t \cos(\theta t - \varepsilon)$$

F.eks. $\gamma = \frac{1}{2}$ $\quad a_1 = -\frac{1}{2}(1+1) = -1 \quad a_2 = \gamma\alpha = \frac{1}{2}$

$D = \frac{1}{2}^2 \cdot 2^2 - 4 \cdot \frac{1}{2} \cdot 1 = 1 - 2 = -1$

$b_1 = \frac{1+i}{2} \quad b_2 = \frac{1-i}{2} \quad h = \frac{1}{2} \quad v = \frac{1}{2}$

$R = \sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} = \sqrt{2 \frac{1}{2}^2} = \sqrt{2}\,\frac{1}{2} < 1$

Da dæmpet oscillerende. Konvergens mod $\frac{G_0}{1-\frac{1}{2}} = 2G_0$