Laplace approximation

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1 Laplace approximation in one dimension

Consider an integral

$$I_n = \int_{\mathbb{R}} \exp(nh(x))g(x)\mathrm{d}x$$

where $h(x)$ is three times differentiable and assume there exists $\hat{x}$ so that

C1 $H = -h''(\hat{x}) > 0$ and $h'(\hat{x}) = 0$

C2 for any $\Delta > 0$ there exists an $\epsilon > 0$ so that $h(\hat{x}) - h(x) > \epsilon$ for $|x - \hat{x}| > \Delta$

C3 there exists a $\delta > 0$ so that $|h^{(3)}(x)| < K$ and $|g(x)| < C$ for $|x - \hat{x}| \leq \delta$

C4 a) $\int_{\mathbb{R}} |g(x)|\mathrm{d}x < K_a$ or b) $\int_{\mathbb{R}} \exp(h(x))|g(x)|\mathrm{d}x < K_b$

Then

$$\frac{I_n}{\exp(nh(\hat{x}))g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}}} \to 1 \tag{1}$$

as $n \to \infty$.

Proof:

Pick a $\delta > 0$ as in C3 and let $A_\delta = [\hat{x} - \delta, \hat{x} + \delta]$. Then using C2 and C4 a),

$$\int_{A_\delta^c} \exp(nh(x) - nh(\hat{x}))g(x)\mathrm{d}x \leq \exp(-n\epsilon)K_a.$$

Using instead C4 b),

$$\begin{aligned}
&\int_{A_\delta^c} \exp(nh(x) - nh(\hat{x}))g(x)\mathrm{d}x \\
=&\int_{A_\delta^c} \exp((n-1)(h(x) - h(\hat{x})) + h(x) - h(\hat{x}))g(x)\mathrm{d}x \\
\leq& \exp(-(n-1)\epsilon)\exp(-h(\hat{x}))K_b.
\end{aligned}$$

Thus

$$\frac{I_n \exp(-nh(\hat{x}))}{g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}}}$$
$$=\frac{\int_{A_\delta} \exp(nh(x)-nh(\hat{x}))g(x)\mathrm{d}x}{g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}}}+\frac{\int_{A_\delta^c} \exp(nh(x)-nh(\hat{x}))g(x)\mathrm{d}x}{g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}}}$$

where, in case of a),

$$\left|\frac{\int_{A_\delta^c} \exp(nh(x)-nh(\hat{x}))g(x)\mathrm{d}x}{g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}}}\right| \leq \frac{\exp(-n\epsilon)K_a}{g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}}} \to 0,$$

and similar in case of b).

We thus need to approximate

$$J_n = \int_{A_\delta} \exp(nh(x)-nh(\hat{x}))g(x)\mathrm{d}x$$

by

$$g(\hat{x})\sqrt{2\pi n^{-1}H^{-1}} = n^{-1/2}\int_{\mathbb{R}} \exp(-\frac{H}{2}u^2)g(\hat{x})\mathrm{d}u.$$

To do this we make the change of variable $u=\sqrt{n}(x-\hat{x})$ whereby

$$J_n = n^{-1/2}\int_{B_{n,\delta}} \exp(nh(n^{-1/2}u+\hat{x})-nh(\hat{x}))g(n^{-1/2}u+\hat{x})\mathrm{d}u$$

where $B_{n,\delta}=[-\sqrt{n}\delta,\sqrt{n}\delta]$. Define

$$f_n(u)=\exp(nh(n^{-1/2}u+\hat{x})-nh(\hat{x}))g(n^{-1/2}u+\hat{x})-\exp(-\frac{H}{2}u^2)g(\hat{x})$$

Then

$$|\int_{B_{n,\delta}} \exp(nh(n^{-1/2}u+\hat{x})-nh(\hat{x}))g(n^{-1/2}u+\hat{x})\mathrm{d}u-\int_{\mathbb{R}}\exp(-\frac{H}{2}u^2)g(\hat{x})\mathrm{d}u|$$
$$=|\int_{B_{n,\delta}}\exp(nh(n^{-1/2}u+\hat{x})-nh(\hat{x}))g(n^{-1/2}u+\hat{x})\mathrm{d}u-\int_{B_{n,\delta}}\exp(-\frac{H}{2}u^2)g(\hat{x})\mathrm{d}u-\int_{B_{n,\delta}^c}\exp(-\frac{H}{2}u^2)g(\hat{x})\mathrm{d}u|$$
$$\leq \int_{B_{n,\delta}}|f_n(u)|\mathrm{d}u+|\int_{B_{n,\delta}^c}\exp(-\frac{H}{2}u^2)g(\hat{x})\mathrm{d}u|$$

where $\lim_{n\to\infty}|\int_{B_{n,\delta}^c}\exp(-\frac{H}{2}u^2)g(\hat{x})\mathrm{d}u|=0$. Hence $\int_{B_{n,\delta}}|f_n(u)|\mathrm{d}u\to 0$ will imply

$$\frac{J_n}{g(\hat{x})\sqrt{2\pi H^{-1}n^{-1}}}=\frac{n^{1/2}J_n}{g(\hat{x})\sqrt{2\pi H^{-1}}}\to 1.$$

We expand $nh(n^{-1/2}u+\hat{x})-nh(\hat{x})$ using a second order Taylor expansion around $u=0$:

$$nh(n^{-1/2}u+\hat{x})-nh(\hat{x})=-\frac{H}{2}u^2+R_n(u)$$

where

$$R_n(u)=nn^{-3/2}\frac{1}{6}h^{(3)}(c)u^3=\frac{n^{-1/2}}{6}h^{(3)}(c)u^3$$

and c is between $\hat{x}$ and $n^{-1/2}u+\hat{x}$. Now, $u\in B_{n,\delta}$ implies $h^{(3)}(c)\leq K$ so $R_n(u)$ tends to zero for any fixed $u\in\mathbb{R}$ meaning that

$$\begin{aligned}&1[u\in B_{n,\delta}]f_n(u)\\=&1[u\in B_{n,\delta}]|\exp(nh(n^{-1/2}u+\hat{x})-nh(\hat{x}))g(n^{-1/2}u+\hat{x})-\exp(-\frac{H}{2}u^2)g(\hat{x})|\end{aligned}\tag{2}$$

converges to zero for any $u\in\mathbb{R}$. Also

$$|R_n(u)|\leq\frac{n^{-1/2}}{6}Kn^{1/2}\delta u^2=\frac{1}{6}K\delta u^2$$

so

$$1[u\in B_{n,\delta}]\exp(nh(n^{-1/2}u+\hat{x})-nh(\hat{x}))|g(n^{-1/2}u+\hat{x})|\leq\exp(-\frac{H}{2}u^2+\frac{1}{6}K\delta u^2)C$$

where the upper bound is integrable as a function of u for δ small enough. We can now use the triangle inequality to show that (2) is bounded by an integrable function. Hence $\int_{B_{n,\delta}}|f_n(u)|\mathrm{d}u\to 0$ follows by dominated convergence.

2 Order of approximation

Here we assume for simplicity that $g(x)=1$ (meaning that condition C4 b) is relevant). We also assume $h^{(4)}(x)<\tilde{K}$ for all $|x-\hat{x}|\leq\delta$. We next employ a fourth order Taylor expansion

$$nh(n^{-1/2}u+\hat{x})-nh(\hat{x})=-\frac{H}{2}u^2+n^{-1/2}\frac{H_3}{6}u^3+\tilde{R}_n(u)$$

where $H_3=h^{(3)}(\hat{x})$ and

$$\tilde{R}_n(u)=n^{-1}\frac{H_4(u)}{24}u^4$$

where $H_4(u)=h^{(4)}(\tilde{c})$ and $\tilde{c}$ is between $\hat{x}$ and $n^{-1/2}u+\hat{x}$. A first order Taylor expansion of $\exp(\cdot)$ yields

$$\exp(n^{-1/2}\frac{H_3}{6}u^3+\tilde{R}_n(u))=1+n^{-1/2}\frac{H_3}{6}u^3+\tilde{R}_n(u)+\frac{\exp(b)}{24}\left(n^{-1/2}\frac{H_3}{6}u^3+\tilde{R}_n(u)\right)^2$$

where b is between 0 and $n^{-1/2}\frac{H_3}{6}u^3 + \tilde{R}_n(u)$.

We now assess

$$n^{1/2}J_n = \int_{B_{n,\delta}} \exp(-\frac{H}{2}u^2)\left[1 + n^{-1/2}\frac{H_3}{6}u^3 + \tilde{R}_n(u) + \frac{\exp(b)}{24}\left(n^{-1/2}\frac{H_3}{6}u^3 + \tilde{R}_n(u)\right)^2\right]\mathrm{d}u$$

by considering each term inside the integral separately (and using dominated convergence to replace $B_{n,\delta}$ by $\mathbb{R}$):

$$\lim_{n\to\infty}\int_{B_{n,\delta}} \exp(-\frac{H}{2}u^2)\mathrm{d}u = \int_{\mathbb{R}} \exp(-\frac{H}{2}u^2)\mathrm{d}u = \sqrt{2\pi H^{-1}}$$

For the second term times n we have

$$\lim_{n\to\infty} n^{1/2}\int_{B_{n,\delta}} \exp(-\frac{H}{2}u^2)\frac{H_3}{6}u^3\mathrm{d}u = 0$$

since $\exp(-\frac{H}{2}(-u)^2)\frac{H_3}{6}(-u)^3 = -\exp(-\frac{H}{2}u^2)\frac{H_3}{6}u^3$. For the third term multiplied by n, we get

$$\lim_{n\to\infty}\int_{B_{n,\delta}} \exp(-\frac{H}{2}u^2)n|\tilde{R}_n(u)|\mathrm{d}u \le \int_{\mathbb{R}} \exp(-\frac{H}{2}u^2)\frac{\tilde{K}}{24}u^4\mathrm{d}u < \infty.$$

For the last term,

$$\int_{B_{n,\delta}} \exp(-\frac{H}{2}u^2)\exp(b)\left(n^{-1/2}\frac{H_3}{6}u^3 + \tilde{R}_n(u)\right)^2\mathrm{d}u$$

we use

$$|n^{-1/2}\frac{H_3}{6}u^3| \le \frac{H_3}{6}\delta u^2 \quad \text{and} \quad |\tilde{R}_n(u)| \le \frac{\tilde{K}}{24}\delta^2 u^2$$

so that $\exp(-\frac{H}{2}u^2 + b)$ becomes dominated by an unnormalized Gaussian density for δ small enough. Further

$$\left(n^{-1/2}\frac{H_3}{6}u^3 + \tilde{R}_n(u)\right)^2 \le n^{-1}\frac{H_3^2}{36}u^6 + n^{-2}\frac{\tilde{K}^2}{24^2}u^8 + n^{-3/2}\frac{|H_3|\tilde{K}}{144}|u|^7.$$

The last term thus becomes $O(n^{-1})$. In conclusion,

$$\frac{n^{1/2}J_n}{\sqrt{2\pi H^{-1}}} = 1 + O(n^{-1}).$$

This further implies for the relative error,

$$\frac{n^{1/2}J_n - \sqrt{2\pi H^{-1}}}{\sqrt{2\pi H^{-1}}} = O(n^{-1})$$

and

$$\frac{n^{1/2}J_n - \sqrt{2\pi H^{-1}}}{n^{1/2}J_n} = \frac{n^{1/2}J_n - \sqrt{2\pi H^{-1}}}{\sqrt{2\pi H^{-1}}}\frac{\sqrt{2\pi H^{-1}}}{n^{1/2}J_n} = O(n^{-1}).$$

3 Tail moments for a Gaussian random variable

In this section we derive a bound for the integral

$$\int_x^\infty u^k \exp(-u^2/2)\mathrm{d}u$$

for a large x. Note first that $u^k < \exp(\alpha u^2/2)$ for any $\alpha > 0$ when u is large enough. Hence for large enough x,

$$\int_x^\infty u^k \exp(-u^2/2\mathrm{d}u) \leq \int_x^\infty \exp(-u^2(1-\alpha)/2)\mathrm{d}u \leq \int_x^\infty \frac{u}{x}\exp(-u^2(1-\alpha)/2)\mathrm{d}u$$

Using the substitution $v = u^2/2$ and choosing $\alpha < 1$ we obtain

$$\begin{aligned}&\int_x^\infty \frac{u}{x}\exp(-u^2(1-\alpha)/2)\mathrm{d}u = \int_{x^2/2}^\infty \frac{1}{x}\exp(-v(1-\alpha))\mathrm{d}v\\ =&[-\frac{\exp(-v(1-\alpha))}{x(1-\alpha)}]_{x^2/2}^\infty = \frac{\exp(-x^2(1-\alpha)/2)}{x(1-\alpha)}.\end{aligned}$$

We used this for the second term in the previous section. For large enough n,

$$n^{1/2}\int_{\sqrt{n}\delta}^\infty u^k \exp(-Hu^2/2)\mathrm{d}u \leq n^{1/2}\frac{\exp(-n\delta^2 H(1-\alpha)/2)}{n^{1/2}\delta(1-\alpha)} = \frac{\exp(-n\delta^2 H(1-\alpha)/2)}{\delta(1-\alpha)}$$

which tends to zero as $n \to \infty$.