Exercise 6

Posterior is

$$p(x|y) \propto p(y|x)p(x) \propto \exp[-\frac{1}{2\sigma^2}\sum_{i \in V}(y_i - x_i)^2]p(x) \qquad (1)$$

We know $p(x)$ is a pairwise MRF which can be written according to Hammersley-Clifford as

$$p(x) \propto \prod_i \phi_{\{i\}}(x_i) \prod_{i<j} \phi_{\{i,j\}}(x_i, x_j)$$

Thus

$$p(x|y) \propto \prod_i \tilde{\phi}_{\{i\}}(x_i) \prod_{i<j} \phi_{\{i,j\}}(x_i, x_j)$$

where

$$\phi_{\{i\}}(x_i) = \exp(-\frac{1}{2}(y_i - x_i)^2)\phi_{\{i\}}(x_i).$$

Hence according to Hammersley-Clifford, $p(x|y)$ is also a MRF.

Exercise 6 continued

Rewriting in terms of vectors y and x we get

$$p(x|y) \propto \exp(-\frac{1}{2}(y-x)^{\mathsf{T}}(\sigma^{-2}I)(y-x)-\frac{1}{2}(x-\mu)^T(Q+\tau I)(x-\mu))$$
$$\propto \exp(-\frac{1}{2}x^{\mathsf{T}}(Q+(\tau+\sigma^{-2})I)x + x^{\mathsf{T}}(\sigma^{-2}y+(Q+\tau I)\mu))$$
$$= \exp(-\frac{1}{2}x^{\mathsf{T}}Kx + x^{\mathsf{T}}KK^{-1}(\sigma^{-2}y+(Q+\tau I)\mu))$$

where $K = Q + (\tau + \sigma^{-2})I$. Hence, according to hint, $X|Y = y|$ is multivariate normal with precision matrix K and mean vector $\xi = K^{-1}(\sigma^{-2}y + (Q + \tau I)\mu)$.

If $\tau = 0$, $K = Q + \sigma^{-2}I$ is still an invertible matrix and $\xi = K^{-1}\sigma^{-2}y$ does not depend on μ when μ is a constant vector.

Exercise 7

Run code in `gibbs_sampler.R`.

Note how results are very different depending on whether we are below critical value 0.88 (with $\tilde{\beta} = 0.4, 0.7$) or above critical value (with $\beta = 0.9$).

Exercise 9

Use code in `bayesian_ising.R`.

Enjoy the nice reconstructed image given by the posterior mean !

Exercise 10

We saw in Exercise 6 that with $\tau = 0$, the posterior mean was

$$K^{-1}\sigma^{-2}y = \frac{1}{\sigma^2}(Q + \sigma^{-2}I)^{-1}y = (\sigma^2 Q + I)^{-1}y$$

Try out code in `bayesian_GMRF.R`. Note that resulting posterior mean is too smooth along the edges of the image.