Recap
Random variables

Discrete random variable

- Sample space is finite or countable many elements
- The probability function f(x) is often tabulated
- Calculation of probabilities

$$P(a < X < b) = \sum_{a<t<b} f(t)$$

Continuous random variable

- Sample space has infinitely many elements
- The density function f(x) is a continuous function
- Calculation of probabilities

$$P(a < X < b) = \int_a^b f(t)\, dt$$

Variance
Definition

Definition:
Let X be a random variable with probability / density function f(x) and expected value μ. The variance of X is then given

$$\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

if X is discrete, and

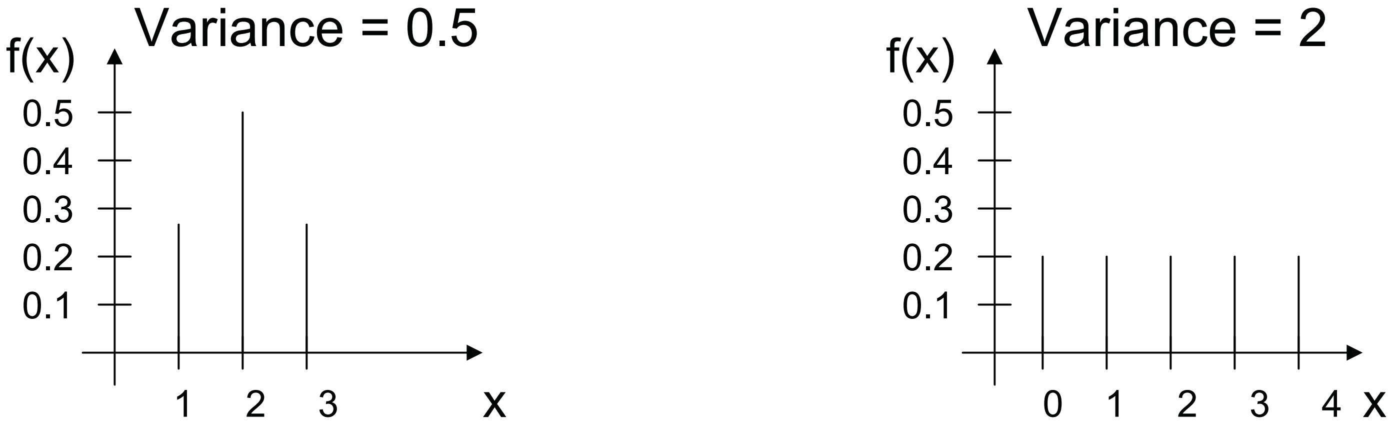
$$\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

if X is continuous.

The standard deviation is the positive root of the variance: $\sigma = \sqrt{\mathrm{Var}(X)}$

Variance
Interpretation

The variance expresses, how dispersed the density / probability function is around the mean.



Rewrite of the variance:

$$\sigma^2 = Var(X) = E[X^2] - \mu^2$$

Variance
Linear combinations

Theorem: Linear combination

Let X be a random variable, and let a and b be constants. For the random variable aX + b the variance is

$$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$$

Examples:

Var (X + 7) = Var (X)
Var (-X) = Var (X)
Var (2X) = 4 Var (X)

Covariance
Definition

Definition:
Let X and Y be to random variables with joint probability / density function f(x,y). **The covariance** between X and Y is

$$\sigma_{XY} = \mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y)$$

if X and Y are discrete, and

$$\sigma_{XY} = \mathrm{Cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy$$

if X and Y are continuous.

Covariance
Interpretation

Covariance between X and Y expresses how X and Y influence each other.

Examples: Covariance between

- X = sale of bicycle and Y = bicycle pumps is positive.
- X = Trips booked to Spain and Y = outdoor temperature is negative.
- X = # eyes on red dice and Y = # eyes on the green dice is zero.

Covariance
Properties

Theorem:
The covariance between two random variables X and Y with means μ_X and μ_Y, respectively, is

$$\sigma_{XY} = \mathrm{Cov}(X,Y) = E[XY] - \mu_X \mu_Y$$

Notice! $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

If X and Y are independent random variables, then

$$\mathrm{Cov}(X,Y) = 0$$

Notice! Cov(X,Y) = 0 does not imply independence!

Variance/Covariance
Linear combinations

Theorem: Linear combination

Let X and Y be random variables, and let a and b be constants.

For the random variables aX + bY the variance is

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab Cov(X, Y)$$

In particular: Var[X+Y] = Var[X] + Var[Y] +2Cov (X,Y)

If X and Y are independent, the variance is

$$Var[X+Y] = Var[X] + Var[Y]$$

Correlation
Definition

Definition:
Let X and Y be two random variables with covariance Cov (X,Y) and standard deviations σ_X and σ_Y, respectively.
The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

It holds that $-1 \le \rho_{XY} \le 1$

If X and Y are independent, then $\rho_{XY} = 0$

Mean, variance, covariance
Collection of rules

Sums and multiplications of constants:

$E(aX) = a\,E(X)$ $Var(aX) = a^2 Var(X)$ $Cov(aX,bY) = ab\,Cov(X,Y)$

$E(aX+b) = aE(X)+b$ $Var(aX+b) = a^2 Var(X)$

Sum:

$E(X+Y) = E(X) + E(Y)$ $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

X and Y are independent: $E(XY) = E(X)\,E(Y)$

$Var(X+Y) = Var(X) + Var(Y)$

Discrete distributions

Four important discrete distributions:

1. The Uniform distribution (discrete)
2. The Binomial distribution
3. The Hyper-geometric distribution
4. The Poisson distribution

Uniform distribution
Definition

Experiment with k equally likely outcomes.

Definition:

Let $X: S \rightarrow R$ be a discrete random variable. If

$$P(X_1 = x_1) = P(X_2 = x_2) = \cdots = P(X_k = x_k) = \frac{1}{k}$$

then the distribution of X is the (discrete) uniform distribution.

Probability function:

$$f(x;k) = \frac{1}{k} \quad for\ x = x_1, x_2, \ldots, x_k$$

(Cumulative) distribution function:

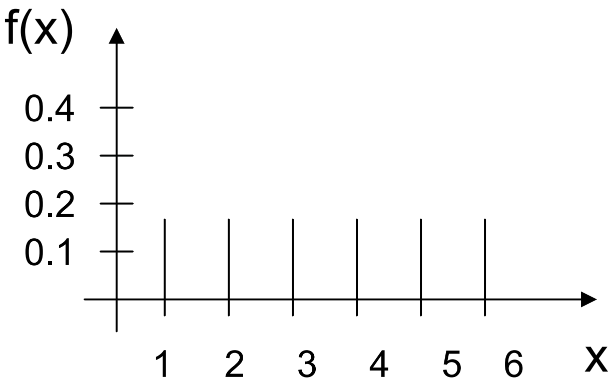
$$F(x;k) = \frac{i}{k} \quad for\ x = x_1, x_2, \ldots, x_k$$

(We assume $x_i < x_{i+1}$)

Uniform distribution
Example

Example: Rolling a dice

X: # eyes



Mean value:

$$E(X) = \frac{1+2+3+4+5+6}{6} = 3.5$$

variance:

$$Var(X) = \frac{(1-3.5)^2 + \ldots + (6-3.5)^2}{6}$$

$$= \frac{35}{12}$$

Probability function:

$$f(x;k) = \frac{1}{6} \quad for\ x = 1,2,\ldots,6$$

Distribution function:

$$F(x;6) = \frac{x}{6} \quad for\ x = 1,2,\ldots,6$$

Uniform distribution
Mean & variance

Theorem:
Let X be a uniformly distributed with outcomes $x_1, x_2, \ldots, x_k$
Then we have

- mean value of X: $$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k}$$
- variance af X: $$Var(X) = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k}$$

Binomial distribution
Bernoulli process

Repeating an experiment with two possible outcomes.

Bernoulli process:

1. The experiment consists in repeating the same trail n times.
2. Each trail has two possible outcomes: “succes” or “failure”, also known as Bernoulli trail.
3. P(”succes”) = p is the same for all trails.
4. The trails are independent.

Binomial distribution
Bernoulli process

Definition:
Let the random variable X be the number of "successes" in the n Bernoulli trails.

The distribution of X is called the binomial distribution.
Notation: $X \sim B(n,p)$

Binomial distribution
Probability & distribution function

Theorem:
If X ~ B(n, p), then X has probability function

$$b(x;n,p) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots,n$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

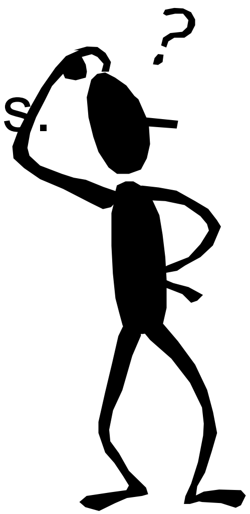
and distribution function

$$B(x;n,p) = P(X \le x) = \sum_{t=0}^{x} b(t;n,p), \quad x = 0,1,2,\ldots,n$$ (See Table A.1)

Binomial distribution
Problem

BILKA has the option to reject a shipment of batteries if they do not fulfil BILKA's "accept policy":

• A sample of 20 batteries is taken: If one or more batteries are defective, the entire shipment is rejected.
• Assume the shipment contains 10% defective batteries.



1. What is the probability that the entire shipment is rejected?

2. What is the probability that at most 3 are defective?

Binomial distribution
Mean & variance

Theorem:
If $X \sim bn(n,p)$, then

- mean of X: $E(X) = np$
- variance of X: $Var(X) = np(1-p)$

Example continued:
What is the expected number of defective batteries?

Hyper-geometric distribution
Hyper-geometric experiment

Hyper-geometric experiment:

1. n elements chosen from N elements without replacement.
2. k of these N elements are ”successes” and N-k are ”failures”

Notice!! Unlike the binomial distribution the selection is done without replacement and the experiments and not independent.

Often used in quality control.

Hyper-geometric distribution
Definition

Definition:
Let the random variable X be the number of "successes" in a hyper-geometric experiment, where n elements are chosen from N elements, of which k are "successes" and N-k are "failures".

The distribution of X is called the hyper-geometric distribution.

Notation: X ~ hg(N,n,k)

Hyper-geometric distribution
Probability & distribution function

Theorem:
If X ~ hg(N, n, k), then X has probability function

$$h(x; N, n, k) = P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, 2, \ldots, n$$

and distribution function

$$H(x; N, n, k) = P(X \le x) = \sum_{t=0}^{x} h(t; N, n, k), \quad x = 0, 1, 2, \ldots, n$$

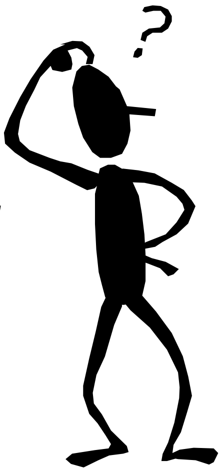
Hyper-geometric distribution
Problem

Føtex receives a shipment of 40 batteries. The shipment is unacceptable if 3 or more batteries are defective.

Sample plan: take 5 batteries. If at least one battery is defective the entire shipment is rejected.

What is the probability of exactly one defective battery, if the shipment contains 3 defective batteries ?

Is this a good sample plan ?



Hyper-geometric distribution
Mean & variance

Theorem:
If X ~ hg(N,n,k), then

- mean of X: $$E(X) = \frac{n\,k}{N}$$
- variance of X: $$Var(X) = \frac{N-n}{N-1}\,n\,\frac{k}{N}\left(1-\frac{k}{N}\right)$$

Poisson distribution
Poisson process

Experiment where events are observed during a time interval.

Poisson process:

1. # events in the interval [a,b] is independent of # events in the interval [c,d], where a<b<c<d } **No memory**
2. Probability of 1 event in a short time interval [a, a + ε] is proportional to ε.
3. The probability of more then 1 event in the short time interval is close to 0.

Poisson distribution
Definition

Definition:
Let the random variable X be the number of events in a time interval of length t from a Poisson process, which has on average λ events pr. unit time.

The distribution of X is called the Poisson distribution with parameter $\mu = \lambda t$.

Notation: $X \sim \text{Pois}(\mu)$, where $\mu = \lambda t$

Poisson distribution
Probability & distribution function

Theorem:

If X ~ Pois(μ), then X has probability function

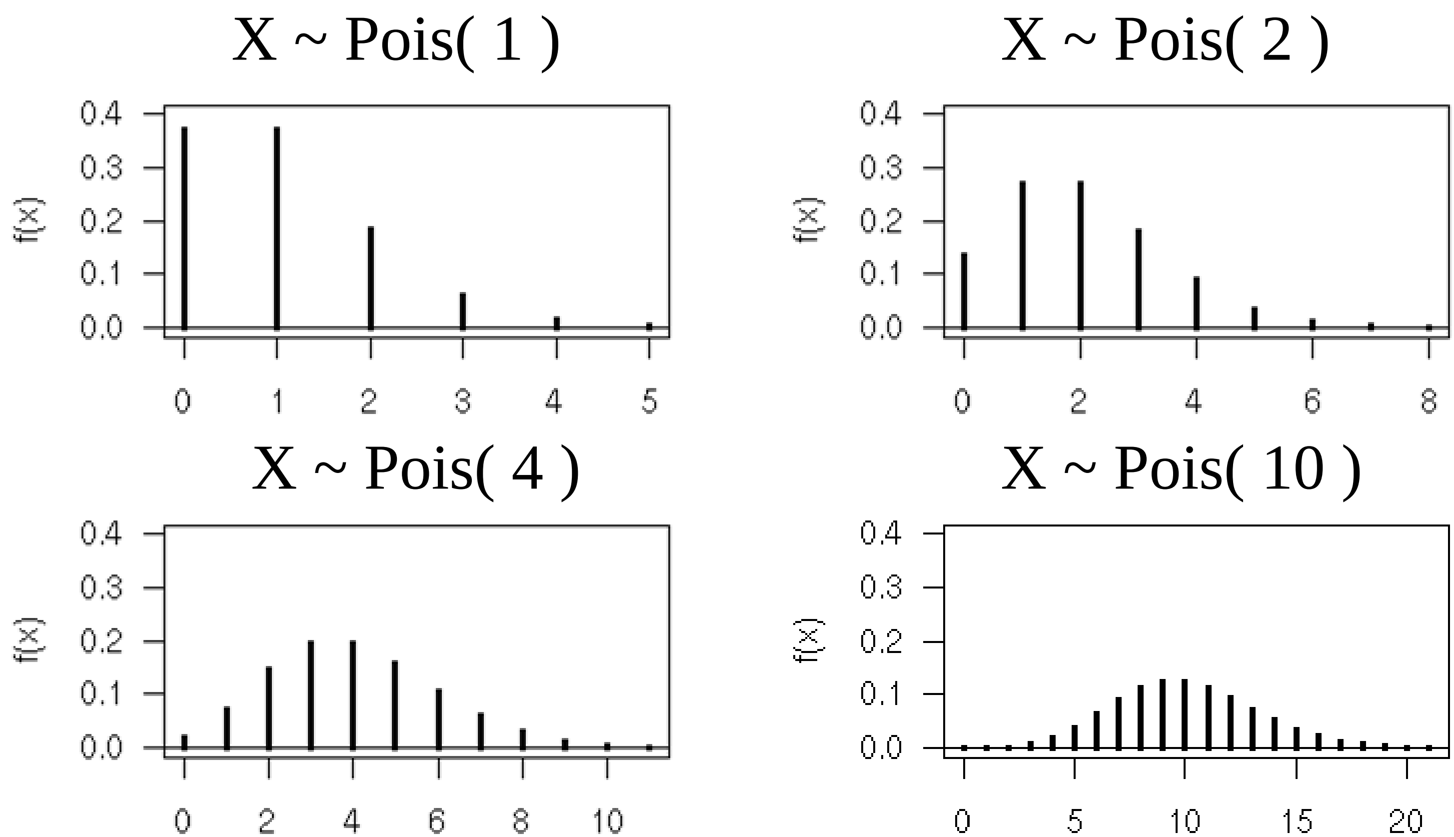
$$p(x\,;\mu) = P(X = x) = \frac{e^{-\mu}\mu^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

and distribution function

$$P(x\,;\mu) = P(X \le x) = \sum_{t=0}^{x} p(t\,;\mu), \quad x = 0, 1, 2, \ldots$$ (see Table A2)

Poisson distribution
Examples

Some examples of $X \sim Pois(\mu)$:



Poisson distribution
Mean & variance

Theorem:
If $X \sim Pois(\mu)$, then

- mean of X: $E(X) = \mu$
- variance of X: $Var(X) = \mu$

Poisson distribution
Problem

Netto have done some research: On weekdays before noon an average of 3 customers pr. minute enter a given shop.

1. What is the probability that exactly 2 customers enter during the time interval 11.38 - 11.39 ?
2. What is the probability that at least 2 customers enter in the same time interval ?
3. What is the probability that at least 10 customers enter the shop in the time interval 10.05 - 10.10 ?